MBS Momentum Testing

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Introduction

- System linear and angular momentums of a joint-connected multibody system (MBS) has two important properties.
  1. Each such momentum is a state of motion corresponding to the generalized coordinates and rates and is independent of the dynamics formalisms. If the dynamics simulation program computes it, then it must be identical to the independently computed momentum state based on the values of the mass, moi's, angular rates, velocities of the bodies and stored momenta that are output from the simulation.
  2. They are constant in the inertial space when no external force or torque is exerted on the system. External forces include jet thruster forces, contact forces, and gravity forces on the bodies. One can allow uniform gravity force over the bodies (i.e. $\boldsymbol{g}$ at each cm is identical) for the momentum test. Momentum conservation compliance reflects the correctness of the equations of motion used in the simulation.
- The following presents the two system momentums of a rigid MBS.

Position and Velocity Representations

- Two practical MBS notations for positions and velocities are:
  - System CM centric notation
  - Joint(1) centric notation

- These are defined in the following charts.

System CM Centric Position Equations

- The inertial position of a body and that of its joint are defined based on the system cm position (see Fig. 1):

$$\bar{r}_i = \hat{r}_i + R$$

$$\bar{\eta}_i = \hat{\eta}_i + R$$

where $\bar{r}_i =$ inertial position of cm_i

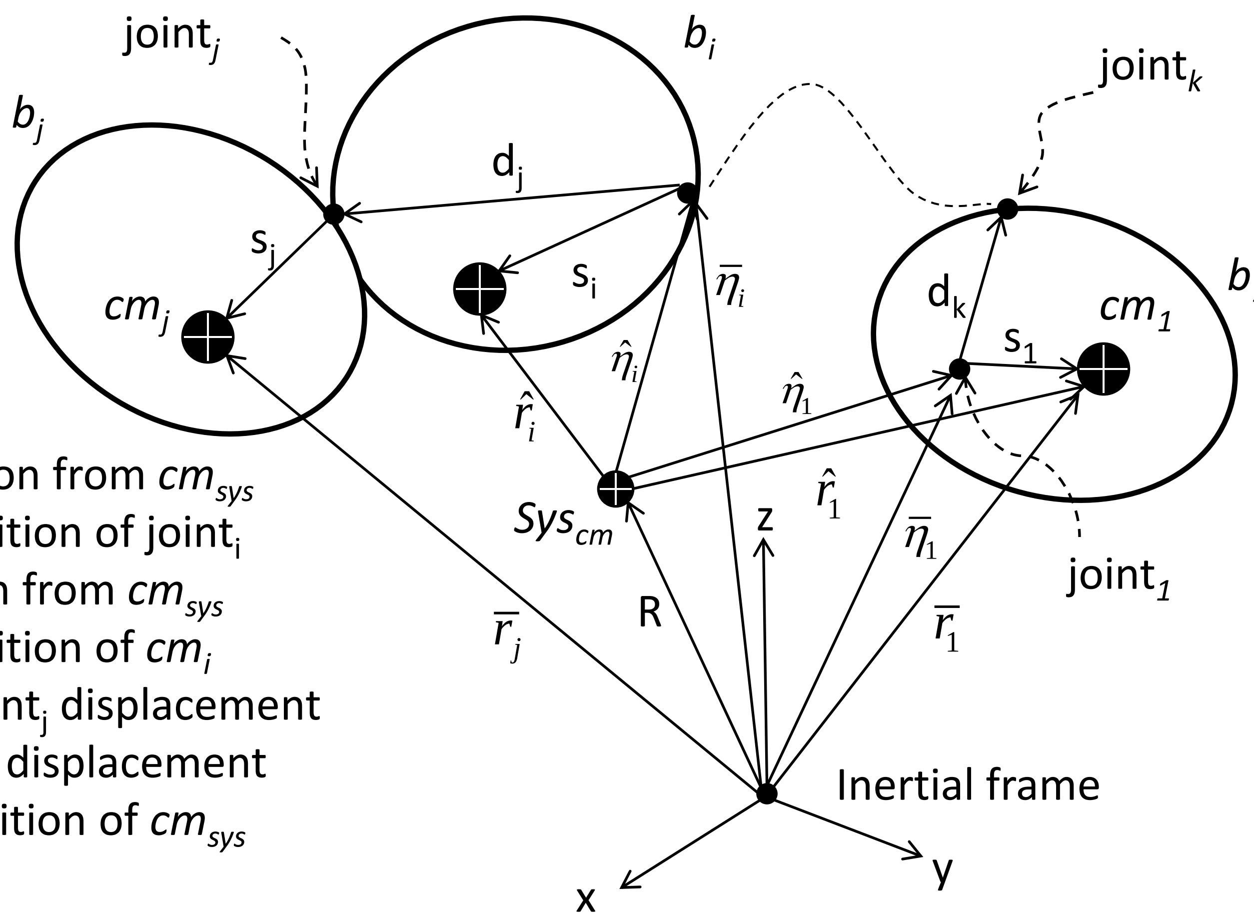
$\bar{\eta}_i =$ inertial position of joint_i

$\hat{r}_i =$ position of cm_i from system cm

$\hat{\eta}_i =$ position of joint_i from system cm

$R =$ inertial position of system cm

Fig. 1 SysCmCentric Notation



b_i $body_i$
$\hat{\eta}_i$ $joint_i$ position from cm_{sys}
$\bar{\eta}_i$ inertial position of $joint_i$
$\hat{r}_i$ cm_i position from cm_{sys}
$\bar{r}_i$ inertial position of cm_i
d_j $joint_{jp}$ to $joint_j$ displacement
s_j $joint_j$ to cm_j displacement
R inertial position of cm_{sys}
b_1 root body
$b_i = b_{jp}$; parent of b_j

Joint (1) Centric Position Equations

- The inertial position of a body and that of its joint are defined based on the joint(1) inertial position (See Fig 2):

$$\bar{r}_i = r_i + \bar{\eta}_1$$

$$\bar{\eta}_i = \eta_i + \bar{\eta}_1$$

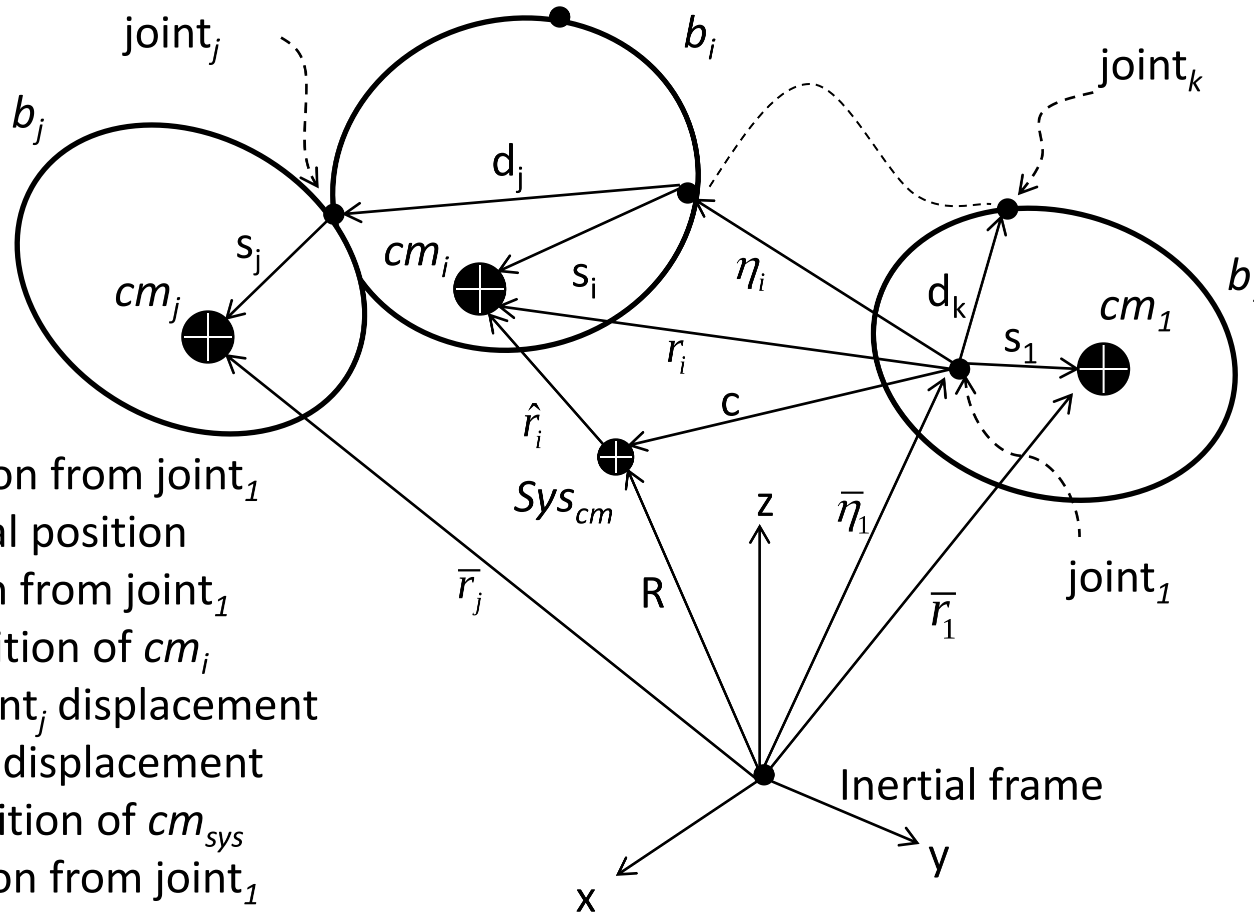
where $\bar{r}_i = \text{inertial position of cm}_i$

$\bar{\eta}_i = \text{inertial position of joint}_i$

$r_i = \text{position of cm}_i \text{ from } \bar{\eta}_1$

$\eta_i = \text{position of joint}_i \text{ from } \bar{\eta}_1 \text{ with } \eta_1 = 0$

Fig. 2 Joint (1) Centric Notation



b_i body$_i$
η_i joint$_i$ position from joint$_1$
$\bar{\eta}_i$ joint$_i$ inertial position
r_i cm_i position from joint$_1$
$\bar{r}_i$ inertial position of cm_i
d_j joint$_{jp}$ to joint$_j$ displacement
s_i joint$_i$ to cm_i displacement
R inertial position of cm_{sys}
c cm_{sys} position from joint$_1$
b_i=b_{jp}; parent of b_j

System Linear and Angular Momenta

- Every MBS has two system momentum states, the linear momentum and the angular momentum of all bodies in the system.

- Both momenta are functions of the system generalized coordinates and rates. They can be computed from the velocities and angular rates of the bodies as shown next.

System Linear Momentum

- The system linear momentum is defined in the inertial frame by

$$P_{sys} = \sum_{i=1:N} m_i \dot{\bar{r}}_i \qquad (1)$$

where, m_i = mass of body b_i

$\dot{\bar{r}}_i$ = inertial velocity of cm of b_i

N = number of bodies in the system

System Angular Momentum

- The system angular momentum is defined in the inertial frame by

$$H_{sys} = \sum_{j=1}^{N} (I_j \omega_j + m_j \hat{r}_j \times \dot{\hat{r}}_j) + \sum_{k=1}^{N_w} h_k^w \qquad (2)$$

where $\omega_j =$ angular rate of b_j in inertial coordinates

$I_j =$ moi of b_j, $m_j =$ mass of b_j

$\hat{r}_j =$ position of cm_j from Sys_{cm} in inertial coordinates

$\dot{\hat{r}}_j =$ velocity of cm_j from Sys_{cm} in inertial coordinates

$h_k^w =$ wheel momentum in inertial coordinates

$N_w =$ number of wheels in the system

Derivation of System Angular Momentum Per SysCmCentric Notation, Fig 1

- The following arguments derive the multibody system angular momentum about the system cm in the inertial frame, Eq. (2), using the System CM Centric Notation of Figure 1.

- If a point, C, is the center of mass of a rigid body b_j then the first mass moment about C is zero, i.e.

$$\int_{b_j} l\,dm = 0 \qquad (3)$$

where dm = differential mass at l from C

- Angular momentum of a b_j about the system center of mass per Fig. 1 is defined by

$$h_j = \int_{b_j} (l + \hat{r}_j) \times (\dot{l} + \dot{\hat{r}}_j + \dot{R})\,dm \qquad (4)$$

where $\hat{r}_j =$ cm of b_j measured from system cm in inertial coordinates

$\dot{\hat{r}}_j = d(\hat{r}_j)/dt$

$\dot{R} =$ inertial velocity of system cm

- Angular momentum of b_j about its own center of mass is

$$\hat{h}_j = \int_{b_j} l \times \dot{l}\, dm = I_j \omega_j \qquad (5)$$

where $\dot{l} = \omega_j \times l$

$\omega_j =$ total angular rate of b_j in inertial coordinates

$I_j = -\int_{b_j} \tilde{l}\,\tilde{l} dm$ in inertial coordinates

$\tilde{l} =$ skew symmetric matrix of l

- Given Eqs. (3,4,5), the angular momentum of b_j about the system center of mass is

$$h_j = I_j \omega_j + m_j \hat{r}_j \times (\dot{\hat{r}}_j + \dot{R}) \qquad (6)$$

where $m_j = \int_{b_j} dm$

- Given Eq.(6), the angular momentum of all *N* bodies in the system about the system center of mass is

$$H_{sys} = \sum_{j=1}^{N} h_j = \sum_{j=1}^{N} (I_j \omega_j + m_j \hat{r}_j \times (\dot{\hat{r}}_j + \dot{R})) \qquad (7)$$

- Equation (7) reduces to

$$H_{sys} = \sum_{j=1}^{N} (I_j \omega_j + m_j \hat{r}_j \times \dot{\hat{r}}_j) \qquad (8)$$

since

$$\sum_{j=1}^{N} m_j \hat{r}_j = 0 \quad \text{, as } \hat{r}_j \text{ is from the system cm, see Fig. 1} \qquad (8a)$$

- Given Eq. (8a) the linear momentum based on $\dot{\hat{r}}_j$ is zero, i.e.

$$p_{sys} = \sum_{j=1}^{N} m_j \dot{\hat{r}}_j = 0 \qquad (9)$$

- Eq. (9) is a constant of motion at all times $\Rightarrow P_{sys} = \left[\sum_{i=1}^{N} m_i \right] \dot{R}$ is constant $\Rightarrow \dot{R}$ is constant if no external forces are applied to the system.

- For systems that have N_w spinning wheels, Eq. 8 becomes

$$H_{sys} = \sum_{j=1}^{N} (I_j \omega_j + m_j \hat{r}_j \times \dot{\hat{r}}_j) + \sum_{k=1}^{N_w} h_k^w \qquad (10)$$

where $h_k^w = g_k^w I_k^w \dot{\theta}_k$; wheel momentum

$g_k^w =$ wheel spin axis in inertial coordinates

$I_k^w =$ wheel spin axis inertia

$\dot{\theta}_k =$ wheel spin speed

- Equation (10) is Eq. (2).

Compute p_{sys} and H_{sys}: Procedure 1

- Equations. (9,10) are functions of $\left\{m_j, I_j, \omega_j, \hat{r}_j, \dot{\hat{r}}_j, h_k^w\right\}$ for $j = 1:N, k = 1:N_w$ in System CM centric notation (Fig. 1).

- $(R, \dot{R})$ should also be an output of the simulation

- If the simulation output $\left\{I_j, \omega_j, \hat{r}_j, \dot{\hat{r}}_j, h_k^w\right\}$ for $j = 1:N, k = 1:N_w$
in inertial coordinates, then use them for Eqs. (9, 10) to computes p_{sys} and H_{sys} .

- Body mass set $\left\{m_j\right\}_{j=1:N}$ can be either from the simulation or user supplied

Other Procedures to Compute p_{sys} and H_{sys}

- Three other procedures to compute H_{sys} are given here when the key kinematics parameters from simulation output are different from those required by Eqs. (9, 10) for the p_{sys} and H_{sys} computation.

Compute p_{sys} and H_{sys}: Procedure 2

- If the simulation output $\left\{\omega_j, \overline{r}_j, \dot{\overline{r}}_j, h_k^w, I_j\right\}$ in inertial coordinates (Fig. 1), then compute p_{sys} and H_{sys} by the following procedure:

1. compute $M = \sum_i m_i$; $R = \dfrac{\sum_{j=1}^{N} m_j \overline{r}_j}{M}$; $\dot{R} = \dfrac{\sum_{j=1}^{N} m_j \dot{\overline{r}}_j}{M}$
2. compute $\hat{r}_j = \overline{r}_j - R,\ \dot{\hat{r}}_j = \dot{\overline{r}}_j - \dot{R}$ for $j = 1:N$
3. use $\left\{m_j, I_j, \omega_j, \hat{r}_j, \dot{\hat{r}}_j, h_k^w\right\}_{j=1:N,\,k=1:N_w}$ for Eq. (9, 10) to compute p_{sys} and H_{sys}

- Body mass set $\left\{m_j\right\}_{j=1:N}$ can be either simulation or user supplied

Compute p_{sys} and H_{sys}: Procedure 3

- If the simulation output $\left\{\omega_j^1, \hat{r}_j^1, \dot{\hat{r}}_j^1, [h_k^w]^1, I_j^1, C_1\right\}$ in b_1 coordinates in System CM centric notation (Fig. 1), then compute p_{sys} and H_{sys} by the following procedure

1. compute $\hat{r}_j = C_1\hat{r}_j^1,\ \dot{\hat{r}}_j = C_1\dot{\hat{r}}_j^1,\ I_j = C_1 I_j^1 C_1^T,\ \omega_j = C_1\omega_j^1$ for $j = 1:N$

   where C_1 = coordinate transformation matrix from b_1 to inertial coords.
2. compute $h_k^w = C_1[h_k^w]^1$ for $k = 1:N_w$
3. use $\left\{m_j, I_j, \omega_j, \hat{r}_j, \dot{\hat{r}}_j, h_k^w\right\}_{j=1:N,k=1:N_w}$ for Eqs. (9, 10) to compute p_{sys} and H_{sys}

Compute p_{sys} and H_{sys}: Procedure 4

- If the simulation output $\left\{\omega_j^1, r_j^1, \dot{r}_j^1, [h_k^w]^1, I_j^1, C_1\right\}$ in b_1 coordinates in Joint(1) Centric notation (Fig. 2), then compute p_{sys} and H_{sys} by the following procedure

1. compute $M = \sum_i m_i \;;\; c^1 = \left(\sum_{j=1}^{N} m_j r_j^1\right) / M \;;\; \dot{c}^1 = \left(\sum_{j=1}^{N} m_j \dot{r}_j^1\right) / M$
2. compute $\hat{r}_j = C_1(r_j^1 - c^1),\ \dot{\hat{r}}_j = C_1(\dot{r}_j^1 - \dot{c}^1),\ I_j = C_1 I_j^1 C_1^T,\ \omega_j = C_1 \omega_j^1$

   for $j = 1 : N$

   $C_1 =$ coordinate transformation matrix from b_1 to inertial coords.
3. compute $h_k^w = C_1 [h_k^w]^1$ for $k = 1 : N_w$
4. use $\left\{m_j, I_j, \omega_j, \hat{r}_j, \dot{\hat{r}}_j, h_k^w\right\}_{j=1:N,\, k=1:N_w}$ for Eqs. (9, 10) to compute p_{sys} and H_{sys}

Summary

- Momentum testing means comparison of independently computed p_{sys} and H_{sys} , i.e. by Eqs. (9, 10) versus those produced by the simulation and the observation of the $(R, \dot{R})$ trajectory under the following conditions.
- Simulation is correctly done dynamically if :

  $p_{sys} = 0$ at all times

- And if when no external forces are present:

  $H_{sys} = \text{constant in inertial(I) coordinates}$

  $\dot{R} \quad = 0 \text{ and } R \text{ is constant in I-coordinates}$

- And if when a uniform gravity force is present on all bodies:

  $H_{sys} = \text{constant in I-coordinates}$

  $\dot{R} \quad = \vec{g}t \text{ and } R \text{ is parabolic I-space}$

- Post-sim Calculation of p_{sys} and H_{sys} of a rigid multibody system by Eqs. (9, 10) requires no knowledge of the system joint-topology and it is valid for any dynamics formulation method used by the simulation.

- Procedures to compute p_{sys} and H_{sys} of a rigid multibody system for 4 types of output from the simulation were presented using Eqs. (9,10).

- p_{sys} , H_{sys} and $(R, \dot{R})$ tests (post-sim) serves to:
  - Identify differences in the momentum state between the ones by the simulation and those independently computed based on the position and rate data output from the simulation
  - check the correctness of mass property and the kinematics eqs. used
  - Verify momentum conservation of the dynamics simulation

- Passing the momentum tests under the stated conditions means that the dynamics principle underlying the simulation and its implementation are credible and the simulation result is useful.

- Also see System Energy tests

- p_{sys} , H_{sys} and $(R, \dot{R})$ tests must be part of any multibody dynamics simulation validation process.