EOM with Newton-Euler Equations

Concurrent Dynamics International

December 2017

Content

- Introduction
- Notations
- Influence Matrices $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}^T$
- O(*N*) Recursive Algorithms
- Acceleration Equations
- Dynamics Equations
- Forward & Inverse Dynamics
- Constraint Forces
- Computing System Mass Matrix
- O(*N*) Solution of EOM
- Examples
- Summary
- References

Introduction

- Equations motion of a N rigid-body system using the Newton-Euler equations is considered. The influence matrix Φ_A and its companion operators are used to derive them. The acceleration solution by both order(N) and order(N^3) methods are presented. Notably, Lagrange multipliers are not needed for the derivations given here.

- Much has been published on the order(N^3) method in the 60's and 70's $\left[6 \text{ to } 10\right]$. The dynamics equations derivation shown here is post 1990's similar to the work by Radriguez-Jain-Kreuze$\left[1\right]$, except for a difference in the definition of the influence matrix operator Φ.

- Tree-structured rigid multibody systems (MBS) that have only single axis rotational motion between joint connected bodies are considered to expedite the discussion. The root body is connected to the ground in all cases considered. These systems encompass robots, mechanical arms, multi-fingered hands, cranes, gimbaled antennas, transmission and suspension systems.

- Generalized coordinates and system rates for these MBS are

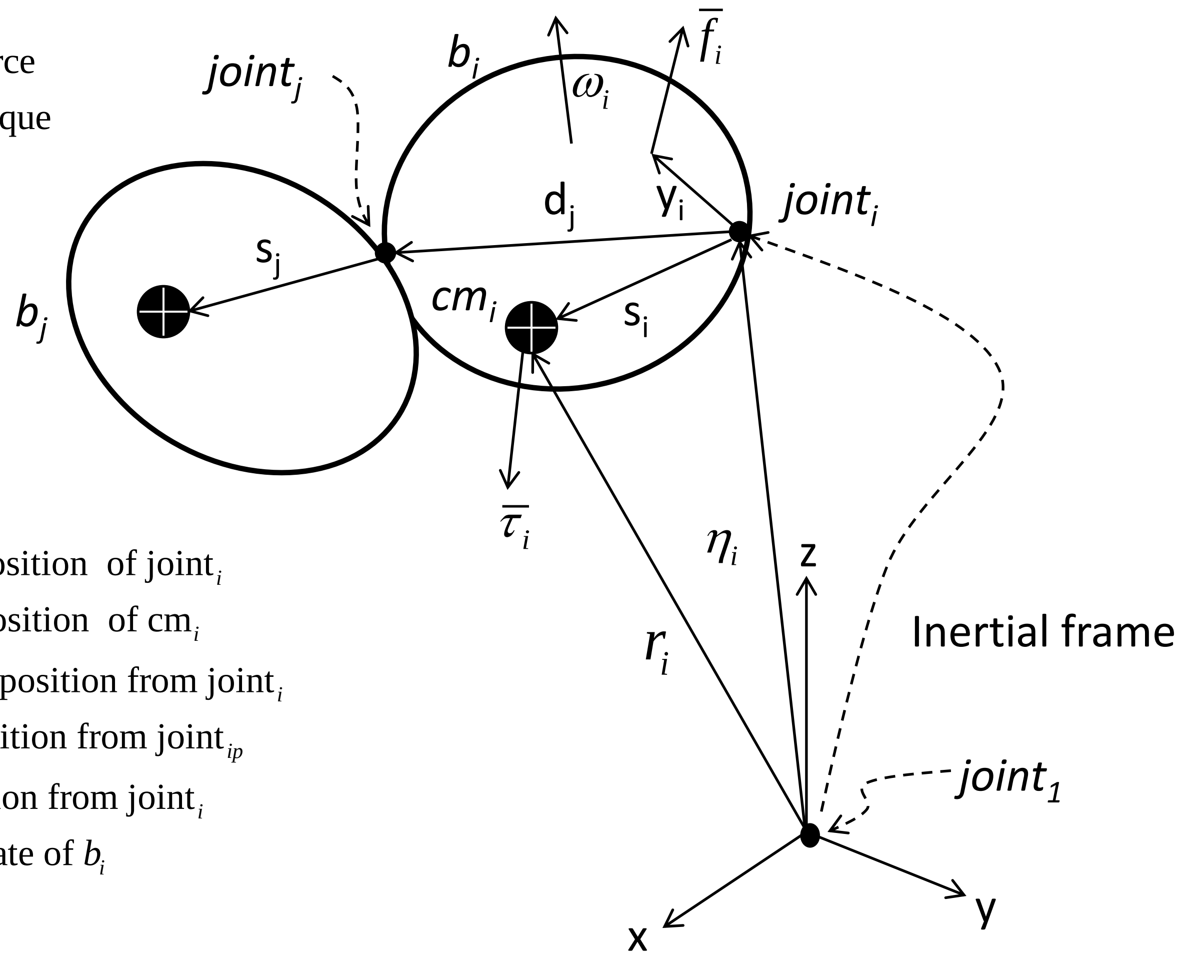
$$q = \{\theta_i\}_{i=1:N}, \ \dot{q} = \{\dot{\theta}_i\}_{i=1:N}$$

where $\theta_i =$ inboard joint angle of b_i

- Ensuing discussions cover 1) dynamics equations of an MBS, 2) solving the dynamics equations for the system accelerations, $\ddot{q}$.

Fig. 1 Notations

$\bar{f}_i$ external force
$\bar{\tau}_i$ external torque



b_i body i

η_i inertial position of $joint_i$

r_i inertial position of cm_i

y_i $\bar{f}_i$ impact position from $joint_i$

d_i $joint_i$ position from $joint_{ip}$

s_i cm_i position from $joint_i$

ω_i angular rate of b_i

- The root body b_1 is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. In the following, b_1 is connected to the ground

- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.

- The chain of bodies between b_1 and b_j shall be denoted as $\{i \mid i \le j\}$ or just $i \le j$.
  The less-than-or-equal relation over body indices is a topological order and not a numerical order.

- The set of bodies branching from b_j shall be denoted as $\{i \mid i \ge j\}$ or just $i \ge j$.
  The greater-than-or-equal relation over body indices is a topological order and not a numerical order.

- All vectors in the following discussion are given in the format x_i^j . The subsript i denotes the body that x belongs to and the superscript j denotes the coordinate frame that the vector is in.

- Vectors with no superscript are given in inertial coordinates unless defined otherwise

- A stacked vector y is a column vector whose elements are also vectors. The latter can be of different sizes. We define this stacked vector as

$$y = [y_1, y_2, y_3, \cdots, y_N], y_i \text{ is a vector}$$

where $\text{size}(y) = len(y) \times 1, \; len(y) = \sum_{j=1}^{N} len(y_j)$

For example:

Let $y_1 = v_x \;, y_2 = \begin{bmatrix} s_x \\ s_y \end{bmatrix} , y_3 = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$

Then $y = [y_1, y_2, y_3] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} v_x \\ s_x \\ s_y \\ w_x \\ w_y \\ w_z \end{bmatrix}$

- Skew matrix notation: $\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$, if $a = \left[a_x, a_y, a_z \right]$

- $e = k \times k$ identity matrix, k depends on context

- The relations $\{ >, \ <, \ \geq, \ \leq \}$ shall mean topological order in the following expressions when used to group bodies, unless stated otherwise.

- $\overline{j} = \{\alpha \mid parent(\alpha) = j\}$, indices of children of b_j

- $ip = parent(i)$, index of parent of b_i

Influence Matrices $\Phi_{\mathcal{A}}$ and $\Phi^T_{\mathcal{A}}$

- Influence matrices , $\Phi_{\mathcal{A}}$ and $\Phi^T_{\mathcal{A}}$, are defined by the parent child relations of the considered mechanism and some block diagonal matrix $\mathcal{A}$:

$$[\Phi_{\mathcal{A}}]_{i,j} = \begin{cases} \mathcal{A}_i \text{ if } \; b_j = \text{parent of } b_i \\ 0_{6\times 6} \text{ otherwise} \end{cases} \qquad \text{Def. 1}$$

$$[\Phi^T_{\mathcal{A}}]_{i,j} = \begin{cases} \mathcal{A}^T_j \text{ if } \; b_i = \text{parent of } b_j \\ 0_{6\times 6} \text{ otherwise} \end{cases} \qquad \text{Def. 2}$$

where $\mathcal{A} = \text{diag}\{\mathcal{A}_j\}_{j=1:N}$, a block diagonal matrix, with $\mathcal{A}_i \in \mathrm{R}^{k\times k}$, $\mathcal{A}_1 = 0$

$k =$ integer > 1

- Forward influence matrix, $\Phi_{\mathcal{A}}$ is strictly lower triangular.
- Backward influence matrix, $\Phi^T_{\mathcal{A}}$ is strictly upper triangular.

- Each body in a tree-configured mechanism has one parent body except for b_1. Thus, each row of $\Phi_{\mathcal{A}}$ has only one non-zero submatrix entry and the first row is all zeros.

- Given that $\Phi_{\mathcal{A}}$ is square and strictly triangular, it is nilpotent of degree *m*, i.e.

$$\Phi_{\mathcal{A}}^{m} = 0$$

where, $m =$ length of the longest link in the system from b_1, and $m \le N$.

- Let $\mathcal{A} = diag\{\mathcal{A}_i\}_{i=1:N}, \mathcal{A}_i \in R^{k\times k}, \mathcal{A}_1 = 0$, and $x = col[x_i],\ z = col[z_i]$, with $x_i, z_i \in R^k, k > 1$.

  Prop1. $z = \Phi_{\mathcal{A}} x$ is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = \mathcal{A}_i x_{ip} \text{ for } i = 1:N \qquad \text{(P1)}$$

  Prop2. $z = \Phi_{\mathcal{A}}^{T} x$ is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j\in\bar{i}} \mathcal{A}_j^{T} x_j \text{ for } i = 1:N \qquad \text{(P2)}$$

- If $\Phi_{\mathcal{A}}$ is square and strictly lower triangular, then $\Psi_{\mathcal{A}} = (e - \Phi_{\mathcal{A}})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}} = e + \Phi_{\mathcal{A}} + \Phi_{\mathcal{A}}^{2} + \cdots + \Phi_{\mathcal{A}}^{m-1} \tag{P3}$$

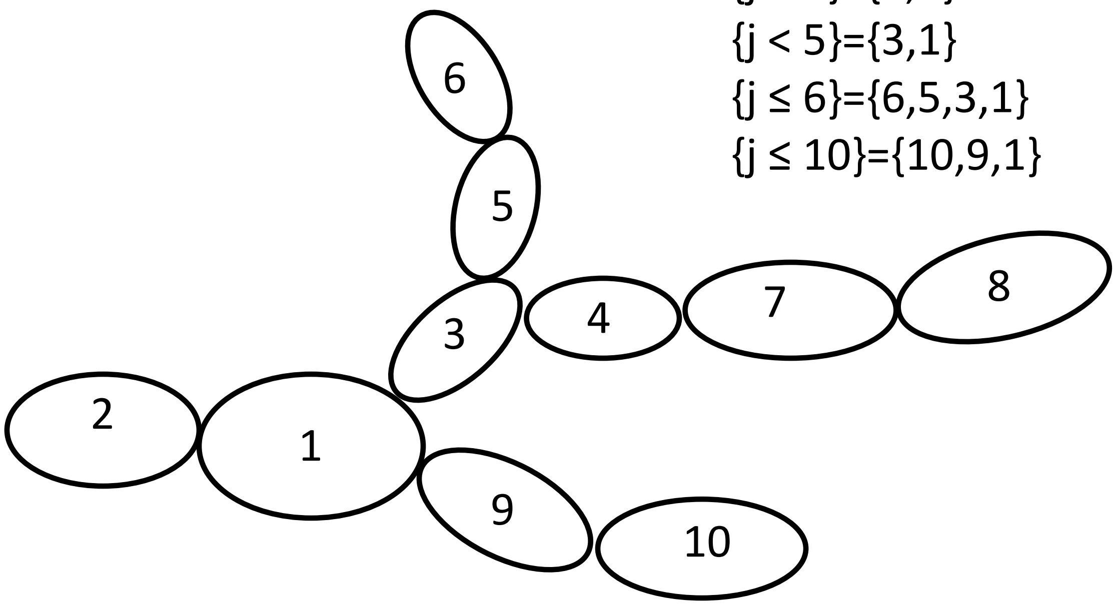
where $m =$ nilpotency of Φ_{A}

- It follows that $\Phi_{\mathcal{A}}^{T}$ is square and strictly upper triangular, and $\Psi_{\mathcal{A}}^{T} = (e - \Phi_{\mathcal{A}}^{T})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}}^{T} = e + \Phi_{\mathcal{A}}^{T} + \Phi_{\mathcal{A}}^{T,2} + \cdots + \Phi_{\mathcal{A}}^{T,m-1} \tag{P4}$$

- The power series expansion of $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}^{T}$ serve to show the existence of $(e + \Phi_{\mathcal{A}})^{-1}$ and $(e + \Phi_{\mathcal{A}}^{T})^{-1}$. More efficient way of computing $\Psi_{\mathcal{A}} x$ and $\Psi_{\mathcal{A}}^{T} x$ for any $x \in \mathrm{R}^{6N}$ is shown shortly.

Fig. 2 Example Body Sets



$\{j \le 2\}=\{2,1\}$
$\{j < 5\}=\{3,1\}$
$\{j \le 6\}=\{6,5,3,1\}$
$\{j \le 10\}=\{10,9,1\}$

$branch_1=\{j \ge 1\}=\{1{:}10\}$
$branch_2=\{j \ge 2\}=\{2\}$
$branch_3=\{j \ge 3\}=\{3{:}8\}$
$branch_4=\{j \ge 4\}=\{4,7,8\}$

$$\overline{1}=\{2,3,9\}$$

$$\overline{2}=\overline{6}=\overline{8}=1\overline{0}=\{empty\}$$

$$\overline{3}=\{4,5\}$$

$$\overline{4}=\{7\}$$

10 body example

Φ_A For Fig. 2 Example

$$\Phi_{\mathcal{A}} = \begin{bmatrix} 0 & & & & & & & & & \\ \mathcal{A}_2 & 0 & & & & & & & & \\ \mathcal{A}_3 & 0 & 0 & & & & -0- & & & \\ 0 & 0 & \mathcal{A}_4 & 0 & & & & & & \\ 0 & 0 & \mathcal{A}_5 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & \mathcal{A}_6 & 0 & & & & \\ 0 & 0 & 0 & \mathcal{A}_7 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_8 & 0 & & \\ \mathcal{A}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{10} & 0 \end{bmatrix}$$

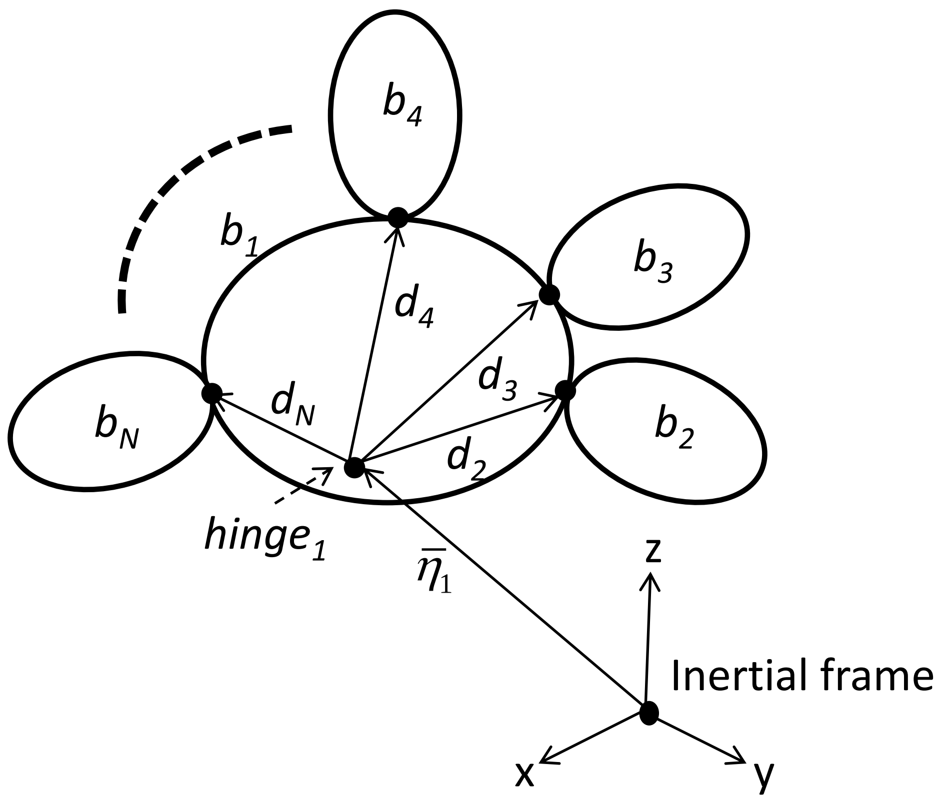
$\Psi_{\mathcal{A}}$ For Fig. 2 Example

If $\mathcal{A}_i$ is a square submatrix $\forall\, i$ for this example, then by (P3)

$$\Psi_{\mathcal{A}} = \left(e - \Phi_{\mathcal{A}}\right)^{-1} = \begin{bmatrix} e & & & & & & & & & \\ \mathcal{A}_2 & e & & & & & & & & \\ \mathcal{A}_3 & 0 & e & & & & -0- & & & \\ \mathcal{A}_4\mathcal{A}_3 & 0 & \mathcal{A}_4 & e & & & & & & \\ \mathcal{A}_5\mathcal{A}_3 & 0 & \mathcal{A}_5 & 0 & e & & & & & \\ \mathcal{A}_6\mathcal{A}_5\mathcal{A}_3 & 0 & \mathcal{A}_6\mathcal{A}_5 & 0 & \mathcal{A}_6 & e & & & & \\ \mathcal{A}_7\mathcal{A}_4\mathcal{A}_3 & 0 & \mathcal{A}_7\mathcal{A}_4 & \mathcal{A}_7 & 0 & 0 & e & & & \\ \mathcal{A}_8\mathcal{A}_7\mathcal{A}_4\mathcal{A}_3 & 0 & \mathcal{A}_8\mathcal{A}_7\mathcal{A}_4 & \mathcal{A}_8\mathcal{A}_7 & 0 & 0 & \mathcal{A}_8 & e & & \\ \mathcal{A}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & \\ \mathcal{A}_{10}\mathcal{A}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{10} & e \end{bmatrix}$$

nilpotency($\Phi_{\mathcal{A}}$)=5

Fig. 3 Nilpotency 2 System



- By Defs. 2 and 3, if $\mathcal{A}$ is a square matrix, then $\Psi_{\mathcal{A}}$ for nilpotency 2 systems per (P3) are

$$\Psi_{\mathcal{A}} = e + \Phi_{\mathcal{A}}$$

with

$$\Phi_{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \mathcal{A}_2 & 0 & 0 & \cdots & 0 \\ \mathcal{A}_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_N & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- A nilpotency 2 tree configured system has all bodies $b_{2:N}$ as immediate children of b_1. See Fig. 3. For these systems the forward recursive operator per (P3) is

$$\Psi_A = \begin{bmatrix} e & 0 & 0 & \cdots & 0 \\ A_2 & e & 0 & \cdots & 0 \\ A_3 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & 0 & 0 & \cdots & e \end{bmatrix}$$

O(N) Recursive Algorithms

- Examples presented earlier showed the lower-triangular nature of $\Psi_{\mathcal{A}}$ given the parent-child relation of bodies in a tree-configured system.

- Computing expressions involving $\left(\Psi_{\mathcal{A}} x\right)$ or $\left(\Psi_{\mathcal{A}}^{T} x\right)$ by direct matrix-vector multiplication (i.e. using (P3) and (P4) for $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}^{T}$) is inefficient.

- The next two algorithms are order(N) computations of those expressions.

Algorithm for $z = \Psi_{\mathcal{A}} x$

- A stacked vector given by $z = \Psi_{\mathcal{A}} x$ is the result of solving $z = \Phi_{\mathcal{A}} z + x$ for z, where $\Phi_{\mathcal{A}}$ is square and strictly lower triangular per Def 1. Thus, given x, the elements of z are obtained by the algorithm below :

  1. set $z_1 = x_1$ since the first row of $\Phi_{\mathcal{A}}$ is zero
  2. for $i = 2 : N$

     $z_i = \mathcal{A}_i z_{ip} + x_i \quad ;$

     end

- The above computation of z given x is order(N). Thus $\Psi_{\mathcal{A}}$ is an order(N) operator.

Algorithm for $z = \Psi_{\mathcal{A}}^T x$

- A stacked vector given by $z = \Psi_{\mathcal{A}}^T x$ is the result of solving $z = \Phi_{\mathcal{A}}^T z + x$ for z, where $\Phi_{\mathcal{A}}^T$ is square and strictly upper triangular. Thus, given x, the elements of z are obtained by the algorithm below :

  1. set $z = x$ since the first column of $\Phi_{\mathcal{A}}^T$ is zero
  2. for $i = N : 2$

     $z_{ip} := z_{ip} + \mathcal{A}_i^T z_i \quad ;$

     end

- The above computation of z given x is order(N). Thus $\Psi_{\mathcal{A}}^T$ is an order(N) operator.

Acceleration Equations

- Relation between $\overline{v}_j$ and v_j is (See Fig. 4)

$$\overline{v}_i = S_i v_i \tag{1}$$

where $\overline{v}_i = [\omega_i, \dot{r}_i]$, $v_i = [\omega_i, \dot{\eta}_i]$ and ω_i = total angular rate of b_i

$\dot{r}_i =$ velocity of cm_i, $\dot{\eta}_i =$ velocity of joint$_i$

$\dot{\eta}_1 = 0$ since joint$_1$ is attached to ground in the considered system

$S_i = \begin{bmatrix} e & 0 \\ -\tilde{s}_i & e \end{bmatrix}$, $s_i =$ displacement from joint$_i$ to cm_i

- Relation between the two accelerations is

$$\dot{\overline{v}}_i = S_i \dot{v}_i + \overline{A}_i \tag{2}$$

where $\overline{A}_i = \begin{bmatrix} 0 \\ \omega_i \times (\omega_i \times s_i) \end{bmatrix}$

- Total accelerations of (2) in stacked column form is

$$\dot{\bar{v}} = S\dot{v} + \bar{A} \qquad (3)$$

where $\dot{\bar{v}} = col[\dot{\bar{v}}_i]_{i=1:N}, S = diag[S_i]_{i=1:N}, \dot{v} = col[\dot{v}_i]_{i=1:N}, \bar{A} = col[\bar{A}_i]_{i=1:N}$

- Relation between v_i and the relative rate $\dot{q}_i$ is

$$v_i = D_i v_{ip} + G_i \dot{q}_i \qquad (4)$$

where $D_i = \begin{bmatrix} e & 0 \\ -\tilde{d}_i & e \end{bmatrix}, D_1 = 0, v_{1p} = 0$

$$G_i = [g_i, 0_{3\times1}] \text{ for } i = 1{:}\ N$$

- Given (4), the relation between the total acceleration $\dot{v}_i$ and the relative acceleration $\ddot{q}_i$ is

$$\dot{v}_i = D_i \dot{v}_{ip} + G_i \ddot{q}_i + A_i \qquad (5)$$

where $A_i = \dot{D}_i v_{ip} + \dot{G}_i \dot{q}_i = \begin{bmatrix} \omega_{ip} \times g_i \dot{\theta}_i \\ \omega_{ip} \times (\omega_{ip} \times d_i) \end{bmatrix}$, $a_1 = 0$ (5a).

- Given Def. 1 and Prop. 1, Eq. (5) can be put in column vector form as

$$\dot{v} = \Phi_D \dot{v} + G\ddot{q} + A \tag{6}$$

where, $\dot{v} = col[\dot{v}_i]_{i=1:N}, D = diag[D_i]_{i=1:N}, G = diag[G_i]_{i=1:N}, \ddot{q} = col[\ddot{q}_i]_{i=1:N}$

$$A = col[A_i]_{i=1:N}$$

- Total acceleration in (6) can be solved to be

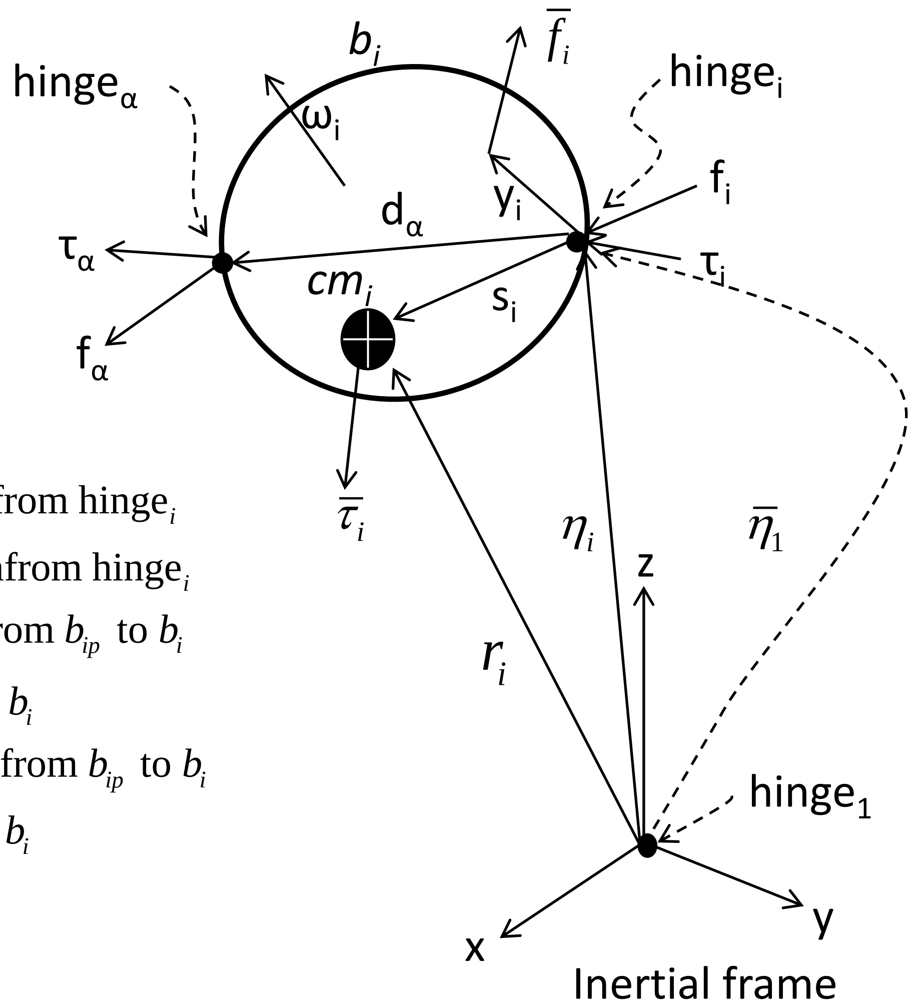
$$\dot{v} = \Psi_D (G\ddot{q} + A) \tag{7}$$

where $\Psi_D = (e - \Phi_D)^{-1}$

- Given (7), Equation (3) becomes

$$\dot{\bar{v}} = S\Psi_D (G\ddot{q} + A) + \bar{A} \tag{8}$$

Fig. 4 Forces and Torque on b_i



- b_i body i
- d_α hinge$_\alpha$ position from hinge$_i$
- y_i $\bar{f}_i$ impact positionfrom hinge$_i$
- f_i interbody force from b_{ip} to b_i
- $\bar{f}_i$ external force on b_i
- τ_i interbody torque from b_{ip} to b_i
- $\bar{\tau}_i$ external force on b_i
- ω_i angular rate of b_i

Dynamics Equations

- Newton-Euler equations for each body b_i of an MBS per Fig. 4 are

$$\left.\begin{aligned}\tau_i - s_i \times f_i = I_i\dot{\omega}_i + \omega_i \times I_i\omega_i - \overline{\tau}_i - (y_i - s_i)\times \overline{f}_i \\ + \sum_{\alpha\in\overline{i}} (\tau_\alpha + (d_\alpha - s_j)\times f_\alpha)\end{aligned}\right\} \qquad (9)$$

$$f_i = m_i\ddot{\mathrm{r}}_i - \overline{f}_i + \sum_{\alpha\in\overline{i}} f_\alpha \qquad (10)$$

- Eqs. (9) and (10) can be made into a 6 x 1 equation to become

$$F_i - \sum_{\alpha\in\overline{i}} D_\alpha^T F_\alpha = S_i^T \overline{M}_i \dot{\overline{v}}_i + \overline{B}_i - Y_i^T \overline{F}_i \qquad (11)$$

where $F_i = \begin{bmatrix}\tau_i \\ f_i\end{bmatrix}, D_\alpha^T = \begin{bmatrix} e & \tilde{d}_\alpha \\ 0 & e\end{bmatrix}, \overline{M}_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i e\end{bmatrix}, \dot{\overline{v}}_i = \begin{bmatrix}\dot{\omega}_i \\ \ddot{r}_i\end{bmatrix}, \overline{B}_i = \begin{bmatrix}\omega_i \times I_i\omega_i \\ 0\end{bmatrix}$

$Y_i^T = \begin{bmatrix} e & \tilde{y}_i \\ 0 & e\end{bmatrix}, \overline{F}_i = \begin{bmatrix}\overline{\tau}_i \\ \overline{f}_i\end{bmatrix}$ (see Fig. 4 for def. of variables.)

$$F_i = \begin{bmatrix} \tau_i \\ f_i \end{bmatrix}, \text{ joint torque-force pair}$$

$$D_\alpha^T = \begin{bmatrix} e & \tilde{d}_\alpha \\ 0 & e \end{bmatrix}, \text{ parent-to-child influence matrix}$$

$$\bar{M}_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i e \end{bmatrix}, \text{ cm centered mass matrix}$$

$$\dot{\bar{v}}_i = \begin{bmatrix} \dot{\omega}_i \\ \ddot{r}_i \end{bmatrix}, \text{ acceleration pair}$$

$$\bar{B}_i = \begin{bmatrix} \omega_i \times I_i \omega_i \\ 0 \end{bmatrix}, \text{ nonlinear force pair}$$

$$Y_j^T = \begin{bmatrix} e & \tilde{y}_i \\ 0 & e \end{bmatrix}, \text{ backward moment-force influence matrix}$$

$$\bar{F}_j = \begin{bmatrix} \bar{\tau}_i \\ \bar{f}_i \end{bmatrix}, \text{ external torque-force pair}$$

- $\mathrm{f}_i = G_i^T F_i$ is the actuation force/torque from b_{ip} to b_i along the moving joint axis. The complement of that force in F_i are the constraint forces/torque that prevent motion along the constrained axes of joint(i).

- Given Def. 2 and Prop. 2, (11) can be put in column vector form as

$$F - \Phi_D^T F = S^T \bar{M}\dot{\bar{v}} + \bar{B} - Y^T \bar{F} \tag{12}$$

where $F = col[F_i]_{i=1:N}, D = diag[D_i]_{i=1:N}, S^T = diag[S_i^T]_{i=1:N}, \bar{M} = diag[\bar{M}_i]_{i=1:N}$,

$$\dot{\bar{v}} = col[\dot{\bar{v}}_i]_{i=1:N}, \bar{B} = col[\bar{B}_i]_{i=1:N}, Y^T = diag[Y_i^T]_{i=1:N}, \bar{F} = col[\bar{F}_i]_{i=1:N}$$

- F in (12) is solved to be

$$F = \Psi_D^T (S^T \bar{M}\dot{\bar{v}} + \bar{B} - Y^T \bar{F}) \tag{13}$$

where $\Psi_D^T = (e - \Phi_D^T)^{-1}$

- Use (8) in (13) to get

$$F = \Psi_D^T (S^T \bar{M} (S\Psi_D (G\ddot{q} + A) + \bar{A}\ ) + \bar{B} - Y^T \bar{F}) \tag{14}$$

- After some rearrangement, (14) becomes

$$F = \Psi_D^T M \Psi_D G\ddot{q} + \Psi_D^T (M \Psi_D A + B - Y^T \bar{F}) \tag{15}$$

where

$$\left.\begin{aligned} M &= S^T \bar{M} S = diag\{S_i^T \bar{M}_i S_i\} = diag\left\{M_i\right\}_{i=1:N} \\ M_i &= \begin{bmatrix} I_i - m_i \tilde{s}_i \tilde{s}_i & m_i \tilde{s}_i \\ -m_i \tilde{s}_i & m_i e \end{bmatrix}, \text{ joint centered mass matrix} \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} B &= S^T \bar{M} \bar{A} + \bar{B} = col\left\{B_i\right\}_{i=1:N} \\ B_i &= \begin{bmatrix} \omega_i \times (I_i - m_i \tilde{s}_i \tilde{s}_i)\omega_i \\ m_i \omega_i \times (\omega_i \times s_i) \end{bmatrix}, \text{ nonlinear force} \end{aligned}\right\} \tag{17}$$

- By projecting F of (15) onto the free joint axes, the dynamics equations becomes

$$\mathcal{M}\ddot{q} = \mathrm{f} - C(\theta,\dot{\theta}) \tag{18}$$

where $\mathrm{f} = G^T F$, actuation force/torque along free axes (18a)

$$C(\theta,\dot{\theta}) = G^T \Psi_D^T (M \Psi_D A + B - Y^T \bar{F}) \tag{18b}$$

$\mathcal{M} = G^T \Psi_D^T M \Psi_D G$, system mass matrix (18c)

- Acceleration state vector in (18) can be solved to be

$$\ddot{q} = \mathcal{M}^{-1}(\mathrm{f} - C(\theta, \dot{\theta})) \qquad (19a)$$

$$= [e - K^T \Phi_D \Psi_\beta G] R^{-1} [e - G^T \Psi_\beta^T \Phi_D^T K](\mathrm{f} - C(\theta, \dot{\theta})) \qquad (19b)$$

(See [1][2] on the factorization of $\mathcal{M}^{-1}$. Note that the $\mathcal{E}$ operator in [2] is the Φ operator here.)

where $\ddot{q} = [\ddot{\theta}_i]_{i=1:N}$

$$\beta = diag[\left(e - G_i K_i^T\right) D_i]_{i=1:N}$$

$$D = diag[D_i]_{i=1:N}, \text{ with } D_1 = 0_{6\times 6}$$

$$R^{-1} = diag[R_i^{-1}]_{i=1:N}, R_i = G_i^T J_i G_i$$

$$J_i = \ branch_i \text{ inertia tensor}$$

$$K = diag[K_i]_{i=1:N}, K_i = J_i G_i R_i^{-1}$$

$$G = diag[G_i]_{i=1:N}$$

- Backward recursive computation of branch inertia tensors $\{J_i\}_{i=1:N}$:

  a) for $i = 1:N$; $J_i = M_i$; end

  b) for $i = N:2$

  $$\hat{J}_i = J_i - J_i G_i R_i^{-1} G_i^T J_i$$

  $$J_{ip} = J_{ip} + D_i^T \hat{J}_i D_i$$

  end

  (see Ref. [1] and [2].)

- Note that (18a) follows from the definition of f. Each f_i of f is the net force/torque from the control and the spring/damper forces transmitted along the free axis of joint_i.

- Eqs. $(19a)$ and $(19b)$[2] are two acceleration state solution options.

- Eq. $(19a)$ requires the assembly of $\mathcal{M}$ using $(18c)$. While that matrix is fractored is an advantage in the assembly process. However, acceleration solution of (18) by any linear equation solvers involving $\mathcal{M}$ is an order(N^3) process.

- Eq. (19b) requires that ($\mathrm{f} - C(\theta,\dot{\theta})$) be operated from right to left using the indicated operators. Every one of those serial operations is an O(N) process. Therefore, the solution process by (19b) is O(N).

Forward & Inverse Dynamics

- Forward dynamics regarding (18) is the problem of solving for the system accelerations $\ddot{q}$ given the interbody forces along the free joint axes and the external forces, i.e. $\{\mathrm{f}_i, \bar{F}_i\}$. This is effectively (19a, or 19b).

- Inverse dynamics regarding (18) is the calculation of the required interbody forces along the joint free axes $\{\mathrm{f}_i\}$ given the system accelerations and the external forces, i.e. $\{\ddot{q}_i^*, \bar{F}_i\}$.

Constraint Forces

- Given that the system accelerations are solved by $\left(19\text{a, or } 19\text{b}\right)$, then all the interbody forces F_i can be determined by $\left(15\right)$.

- Proposition 5. Given that $\mathrm{f}_i = G_i^T F_i$ is the actuation force/torque along the free axis of b_i's inboard hinge, then $\mathrm{f}_i^c = \hat{G}_i^c F_i$ is the constraint forces/torque at the same hinge, where

$$\hat{G}_i \in \mathrm{R}^{6\times 5}, \text{constraint matrix; } G_i^T \hat{G}_i = 0_{1\times 5}, \quad G_i^T G_i = 1, \quad \hat{G}_i^T \hat{G}_i = e_{5\times 5} \qquad (20)$$

More specifically, if $\overline{G}_i^c$ is a particular column of $\hat{G}_i$, then $\overline{\mathrm{f}}_i^c = \overline{G}_i^{cT} F_i$ is the constraint torque (or force) along $\overline{G}_i^c$.

- Example: Let $x\text{-}axis$ be the free axis of joint_i frame and the request is for the constraint force along the $z\text{-}axis$ of that frame. Given that F_i has been computed by (15), then

$$G_i = \left[e_i^x, 0_{3\times1} \right], \ \bar{G}_i^c = \left[0_{3\times1}, e_i^z \right]$$

Ans: $$\bar{\mathrm{f}}_i^c = \left[0_{3\times1}, e_i^z \right]^T F_i$$

where $e_i^z =$ unit $z\text{-}axis$ of joint_i frame in inertial coordinates.

Computing System Mass Matrix

Procedure to compute $\mathcal{M}$ per (18c):

1. Compute $J = \Psi_D G$ $\Rightarrow$ 2. Compute $Q = MJ$

with $J = [J(:,1)\, J(:,2) \cdots J(:,N)]$

$G = [G(:,1)\, G(:,2) \cdots G(:,N)]$

for $i = 1:N$

$J(:,i) = \Psi_D G(:,i)$; per Alg.1

end

2. Compute $Q = MJ$

with $Q = [Q(:,1)\, Q(:,2) \cdots Q(:,N)]$

$M = diag\{M(i), i = 1:N\}$

for $i = 1:N$

$Q(i,:) = M(i) J(i,:)$; row block matrix mult.

end

3. $\mathcal{M} = J^T Q$, end

Q1: what is the order of the above procedure?

Q2: can it be made order(N)?

- Ans1: Steps 1, 2 and 3 are each order(N^2), therefore the process is order(N^2)
- Ans2: No. Computation of $\mathcal{M}$ is order(N^2) per Ans1.

O(N) Solution of EOM

- For $t = \{t_0, t_1, t_2, \cdots t_{end}\}$, do {

    Step 1: Given $[q, \dot{q}]$ at t_k compute $\ddot{q}$:

    a. compute $\{C_i\}$ given $\{q_1, \theta_i\}_{i=1:N} \in q$

    b. compute $\{I_i, g_i, d_i, s_i, r_i, \eta_i, \overline{y}_i, D_i, G_i, M_i, J_i, K_i, R_i, \beta_i\}_{i=1:N}$

    c. compute $\{\omega_i, \dot{r}_i, \dot{\eta}_i\}_{i=1:N}$ given #a, #b and $\dot{q}$

    d. compute $\{\overline{\tau}_i, \overline{f}_i, \mathrm{f}_i\}_{i=1:N}$, application specific input

    e. compute $\{A, B\}$ per (5a) and (17)

    f. solve (18) for $\ddot{q}$

    O(N^3) opt1: compute $(\mathrm{f} - C(\theta, \dot{\theta}) \wedge \mathcal{M})$ per (18a,b,c) and use (19a)

    O(N) opt2: compute $(\mathrm{f} - C(\theta, \dot{\theta}))$ per (18a,b) and use (19b)

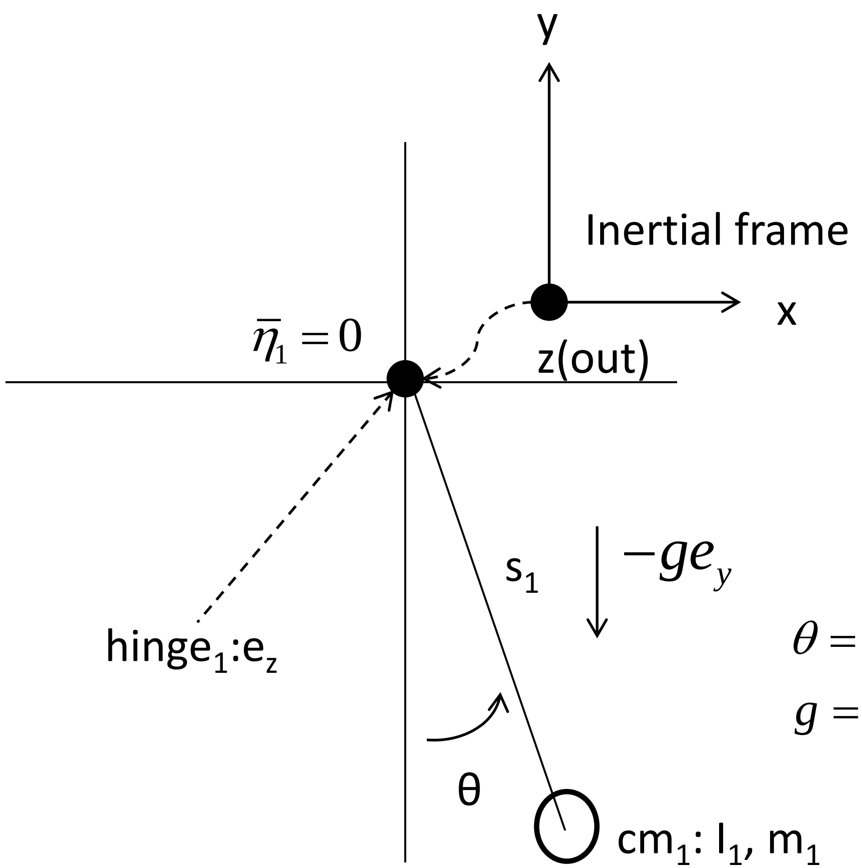
    Step 2: Given $[q, \dot{q}, \ddot{q}]$ at t_k, integrate $[q, \dot{q}]$ to t_{k+1} }

Examples

- The following examples illustrate how to use the methodology presented earlier to derive the eom of specific cases. These derivations can then be compared with other hand derived eom for the same cases for verification.

- It is recommended that the O(N) solution procedure shown earlier be used to develop a generic program that solves all tree-configured N-body dynamics. That program can easily solve the examples presented and produce identical dynamic responses as those produced by solving the hand derived eom.

Example 1
Simple Pendulum



$\theta =$ pendulum angle

$g =$ gravity acceleration

Model Parameters

$$\bar{\eta}_1 = 0, \omega_1 = \dot{\theta} e_z$$

$$s_1 = L(\sin(\theta) e_x - \cos(\theta) e_y) \quad ; \quad \text{cm position from hinge}$$

$$\bar{F}_1 = \begin{bmatrix} 0 \\ -m_1 g_y \end{bmatrix} \quad ; \quad \text{external force}$$

$$\text{f} = \tau = G_1^T F_1 \quad ; \quad \text{hinge torque}$$

$$G = G_1, \text{ with } G_1 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \qquad \text{(M1)}$$

$$\Phi_D = 0_{6\times 6} \qquad \text{(M2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e_{6\times 6} \qquad \text{(M3)}$$

- Let $C(\theta,\dot{\theta}) = G^T \Psi_D^T (M \Psi_D A + B - Y^T \bar{F}\ )$ per (18b) with $A = 0$ and $\Psi_D = e$ for this example.

  Thus, given (17) and (M3), we have

  $$Y^T = \begin{bmatrix} e & \tilde{s}_1 \\ 0 & e \end{bmatrix}$$

  $$B = [0, -m_1 \dot{\theta}^2 s_1]$$

  $$Y^T \bar{F} = [-\tilde{s}_1 m_1 g e_y, -m_1 g e_y]$$

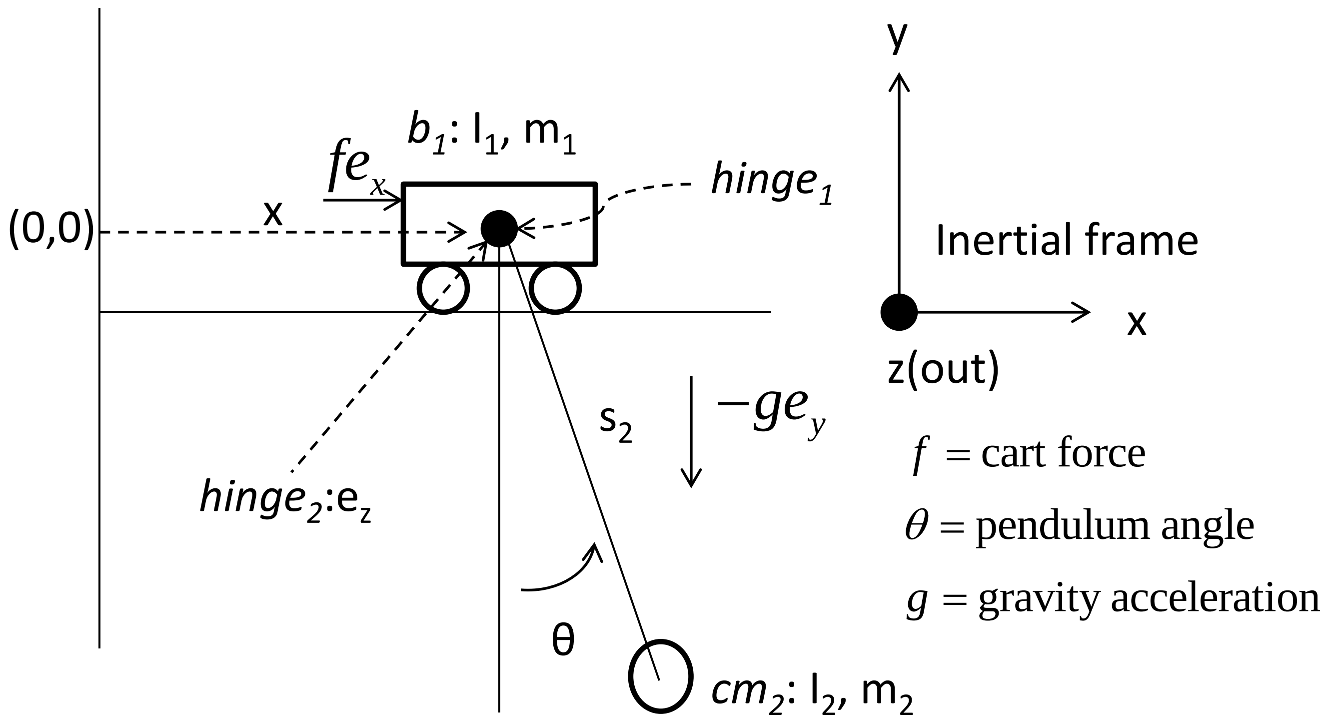
  $$\Psi_D^T (B - Y^T \bar{F}\ ) = [\tilde{s}_1 m_1 g e_y, -m_1 \dot{\theta}^2 s_1 + m_1 g e_y]$$

  $$\mathrm{f} - C(\theta,\dot{\theta}) = \tau - m_1 g L \sin(\theta)$$

- EOM per (18) for this pendulum is

  $$(I_1^{zz} + m_1 L^2)\ddot{\theta} = \tau - m_1 \mathrm{g} L \sin(\theta) \qquad \text{(M4)}$$

Example 2
Pendulum with a Moving Base



f = cart force

θ = pendulum angle

g = gravity acceleration

Model Parameters

$d_2 = 0\ , \overline{\eta}_1 = xe_x, \omega_1 = 0, \omega_2 = \dot{\theta} e_z$

$s_1 = 0, s_2 = L(\sin(\theta)e_x - \cos(\theta)e_y)$; cm position from hinge

$\overline{F}_1 = \begin{bmatrix} 0 \\ fe_x - m_1 ge_y \end{bmatrix}, \overline{F}_2 = \begin{bmatrix} 0 \\ -m_2 ge_y \end{bmatrix}$; external forces

Let $\mathrm{f} = G^T F = [0, 0]$; joint transmitted force/torque

$$G = diag\{G_1, G_2\},\ \text{with } G_1 = \begin{bmatrix} 0 \\ e_x \end{bmatrix}, G_2 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \qquad \text{(N1)}$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, \text{ where } 0 = 0_{6\times 6}\ \text{ nilpotency}(\Phi_D) = 2 \qquad \text{(N2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6\times 6} & 0 \\ e_{6\times 6} & e_{6\times 6} \end{bmatrix} \qquad \text{(N3)}$$

- Let $C(\theta,\dot{\theta}) = G^T \Psi_D^T \left(M \Psi_D A + B - Y^T \overline{F} \right)$ per (18b), and we have $A = 0$ and $\omega_1 = 0$ for this example. Given (17) and (N3), we have

$$B = [[0,0],[0,-m_2 \dot{\theta}^2 s_2]]$$

$$Y_1^T = 0_{6\times 6}, Y_2^T = \begin{bmatrix} e & \tilde{s}_2 \\ 0 & e \end{bmatrix}$$

$$Y^T \overline{F} = [[0,\ fe_x - m_1 g e_y],[-\tilde{s}_2 m_2 g e_y, -m_2 g e_y]\ ]$$

$$\Psi_D^T \left(B - Y^T \overline{F} \right) = [[\tilde{s}_2 m_2 g e_y, -fe_x + m_1 g e_y - m_2 \dot{\theta}^2 s_2 + m_2 g e_y],$$
$$[\tilde{s}_2 m_2 g e_y, m_2 g e_y - m_2 \dot{\theta}^2 s_2]\ ]$$

$$C(\theta,\dot{\theta}) = [-f - m_2 L s(\theta) \dot{\theta}^2, m_2 \mathrm{g} L s(\theta)]$$

Given (18a) and $\mathrm{f} = [0,0]$ for this case,

$$\mathrm{f} - C(\theta,\dot{\theta}) = [\,f + m_2 L s(\theta) \dot{\theta}^2, -m_2 \mathrm{g} L s(\theta)]$$

- EOM for this pendulum on a moving base is

$$\begin{bmatrix} m_1 + m_2 & m_2 L\cos(\theta) \\ m_2 L\cos(\theta) & I_2^{zz} + m_2 L^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f + m_2 L\sin(\theta)\dot{\theta}^2 \\ -m_2 \mathrm{g} L\sin(\theta) \end{bmatrix} \tag{N4}$$

The system mass matrix in (N4) was derived using Eq. (18c)

Summary

- *EOM* for a tree configured multibody system using generalized coordinates and Newton-Euler equations has been presented.

- *EOM* formulation here requires no joint constraints to enforce. One can extract the interbody constraint force or torque at any joint from Eq. (15), once the system accelerations have been determined.

- O(N^3) and O(N) solution to the *EOM,* Eq. (18), have been presented. (See Eq. (19a) and Eq. (19b).)

References

1. Rodriguez, G., Jain, A., Kreutz-Delgado, K., ‘Spatial Operator Algebra for Manipulator Modeling and Control’, The International Journal of Robotics Research, Vol. 10, No. 4, August, 1991.
2. Tong, M. ,’Inverse Mass Matrix Factorization Using Momentum Equations of a Rigid Multibody System’, AIAA Guidance, Navigation and Control Conference, 13-16 August, 2012, Minneapolis, Minnesota, Paper No. 1359569
3. Greenwood, D., ‘Classical Dynamics’, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977.
4. Goldstein, H., ‘Classical Mechanics’, Reading, Mass: Addison-welsley Press, Inc. 1950.
5. Pars, L.A., ‘A Treatise on Analytical Dynamics’, London: Heinemann, 1965.
6. Hooker, W. W., and Margulies, G. 1965. The Dynamical Attitude Equations for an n-Body Satellite. J. Astronautical Sciences, vol. 12, no. 4, pp. 123-128.
7. Likins, P.W., 1975. Point-Connected Rigid Bodies in a Topological Tree. Celestial Mechanics, Vol. 11, No.3, 301-317..
8. Wittenburg, J., 1977. *Dynamics of Systems of Rigid Bodies*. Stuttgart: B. G. Teubner.
9. Ho, J.Y.L, 1977. Direct Path Method for Flexible Multibody Spacecraft Dynamics. Journal of Spacecraft and Rockets, Vol. 14, pp. 102-110.
10. Orlandea, N., Chace, M. A., and Calahan, D. A. 1977. A Sparsity-Oriented Approach to the Dynamic Analysis and Design of Mechanical Systems|Part1. Trans. ASME J. Engineering for Industry, vol. 99, no. 3, pp. 773-779.