

# EOM with Newton-Euler Equations

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# Introduction

- Equations motion of a  $N$  rigid-body system using the Newton-Euler equations is considered. The influence matrix  $\Phi_A$  and its companion operators are used to derive them. The acceleration solution by both order( $N$ ) and order( $N^3$ ) methods are presented. Notably, Lagrange multipliers are not needed for the derivations given here.
- Much has been published on the order( $N^3$ ) method in the 60's and 70's [6 to 10]. The dynamics equations derivation shown here is post 1990's similar to the work by Radriguez-Jain-Kreuze[1], except for a difference in the definition of the influence matrix operator  $\Phi$ .

- Tree-structured rigid multibody systems (*MBS*) that have only single axis rotational motion between joint connected bodies are considered. The root body is connected to the ground. These systems encompass robots, mechanical arms, cranes, gimballed antennas, transmission and suspension systems.

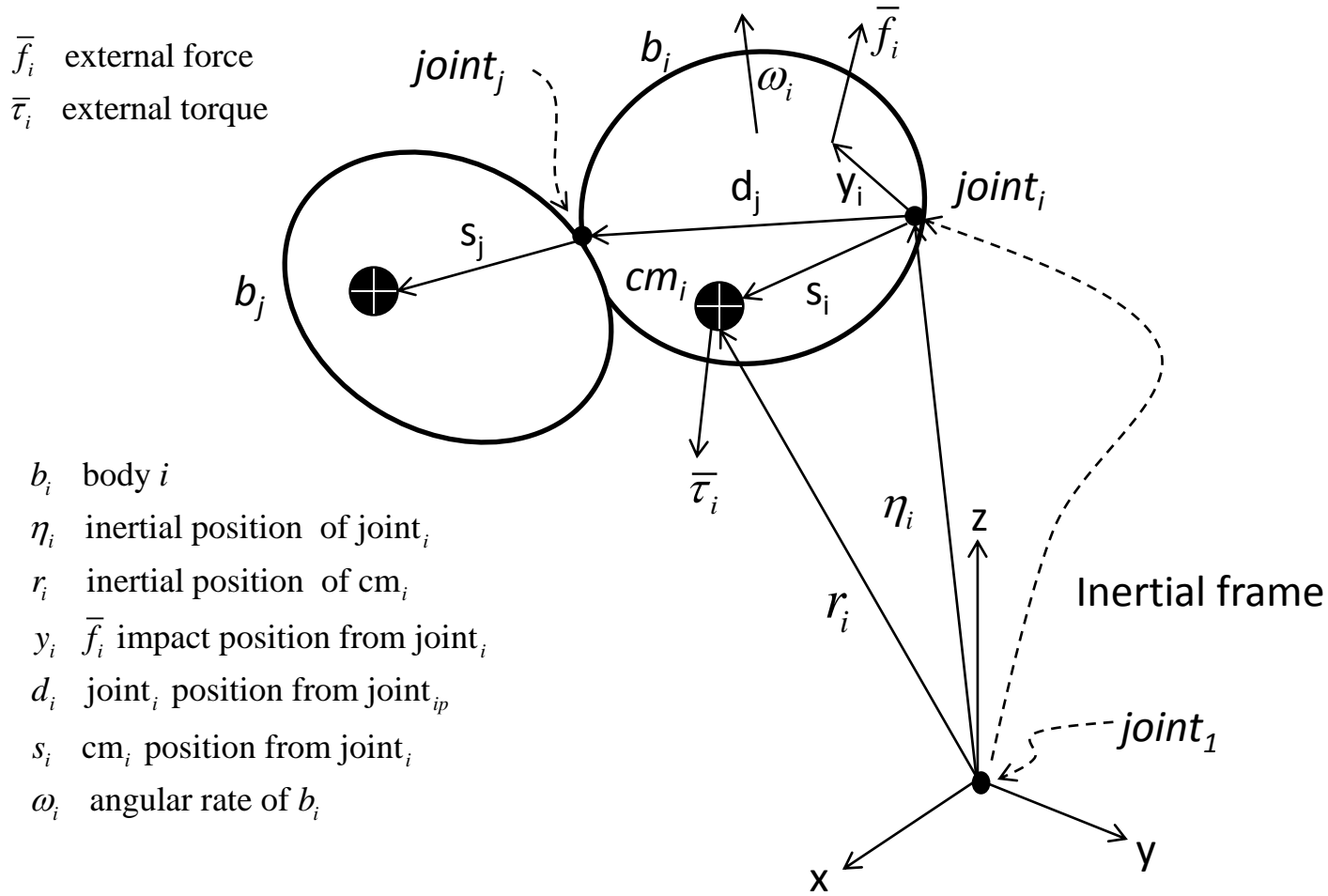
- Generalized coordinates and system rates for these *MBS* are

$$q = \{\theta_i\}_{i=1:N}, \quad \dot{q} = \{\dot{\theta}_i\}_{i=1:N}$$

where  $\theta_i$  = inboard joint angle of  $b_i$

- Ensuing discussions cover 1) dynamics equations of an *MBS*, 2) solving the dynamics equations for the system accelerations,  $\ddot{q}$ .

# Fig. 1 Notations



- The root body  $b_1$  is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. In the following,  $b_1$  is connected to the ground
- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.
- The chain of bodies between  $b_1$  and  $b_j$  shall be denoted as  $\{i \mid i \leq j\}$  or just  $i \leq j$ . The less-than-or-equal relation over body indices is a topological order and not a numerical order.
- The set of bodies branching from  $b_j$  shall be denoted as  $\{i \mid i \geq j\}$  or just  $i \geq j$ . The greater-than-or-equal relation over body indices is a topological order and not a numerical order.
- All vectors in the following discussion are given in the format  $x_i^j$ . The subscript  $i$  denotes the body that  $x$  belongs to and the superscript  $j$  denotes the coordinate frame that the vector is in.
- Vectors with no superscript are given in inertial coordinates unless defined otherwise

- A stacked vector  $y$  is a column vector whose elements are also vectors. The latter can be of different sizes. We define this stacked vector as

$$y = [y_1, y_2, y_3, \dots, y_N], y_i \text{ is a vector}$$

$$\text{where } \text{size}(y) = \text{len}(y) \times 1, \text{len}(y) = \sum_{j=1}^N \text{len}(y_j)$$

For example:

$$\text{Let } y_1 = v_x, y_2 = \begin{bmatrix} s_x \\ s_y \end{bmatrix}, y_3 = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$\text{Then } y = [y_1, y_2, y_3] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} v_x \\ s_x \\ s_y \\ w_x \\ w_y \\ w_z \end{bmatrix}$$

- Skew matrix notation:  $\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$ , if  $a = [a_x, a_y, a_z]$
- $e = k \times k$  identity matrix,  $k$  depends on context
- The relations  $\{ >, <, \geq, \leq \}$  shall mean topological order in the following expressions when used to group bodies, unless stated otherwise.
- $\bar{j} = \{ \alpha \mid \text{parent}(\alpha) = j \}$ , indices of children of  $b_j$



# Influence Matrices $\Phi_A$ and $\Phi_A^T$

- Influence matrices ,  $\Phi_A$  and  $\Phi_A^T$  , are defined by the parent child relations of the considered mechanism and a block diagonal matrix  $A$ :

$$[\Phi_A]_{i,j} = \begin{cases} A_i & \text{if } b_j = \text{parent of } b_i \\ 0_{6 \times 6} & \text{otherwise} \end{cases} \quad \text{Def. 1}$$

$$[\Phi_A^T]_{i,j} = \begin{cases} A_j^T & \text{if } b_i = \text{parent of } b_j \\ 0_{6 \times 6} & \text{otherwise} \end{cases} \quad \text{Def. 2}$$

where  $A = \text{diag}\{A_j\}_{j=1:N}$ , a block diagonal matrix, with  $A_i \in \mathbb{R}^{k \times k}$ ,  $A_1 = 0$   
 $k = \text{integer} > 1$

- Forward influence matrix,  $\Phi_A$  is strictly lower triangular.
- Backward influence matrix,  $\Phi_A^T$  is strictly upper triangular.

- Each body in a tree-configured mechanism has one parent body except for  $b_1$ . Thus, each row of  $\Phi_A$  has only one non-zero submatrix entry and the first row is all zeros.

- Given that  $\Phi_A$  is square and strictly triangular, it is nilpotent of degree  $m$ , i.e.

$$\Phi_A^m = 0$$

where,  $m = \text{length of the longest link in the system from } b_1$ , and  $m \leq N$ .

- Let  $A = \text{diag}\{A_i\}_{i=1:N}$ ,  $A_i \in R^{k \times k}$ ,  $A_1=0$ , and  $x = \text{col}[x_i]$ ,  $z = \text{col}[z_i]$ , with  $x_i, z_i \in R^k$ ,  $k > 1$ .

Prop1.  $z = \Phi_A x$  is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = A_i x_{ip} \text{ for } i = 1:N \quad (\text{P1})$$

Prop2.  $z = \Phi_A^T x$  is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j \in \bar{i}} A_j^T x_j \text{ for } i = 1:N \quad (\text{P2})$$

- If  $\Phi_A$  is square and strictly lower triangular, then  $\Psi_A = (e - \Phi_A)^{-1}$  exists and can be expressed as

$$\Psi_A = e + \Phi_A + \Phi_A^2 + \dots + \Phi_A^{m-1}$$

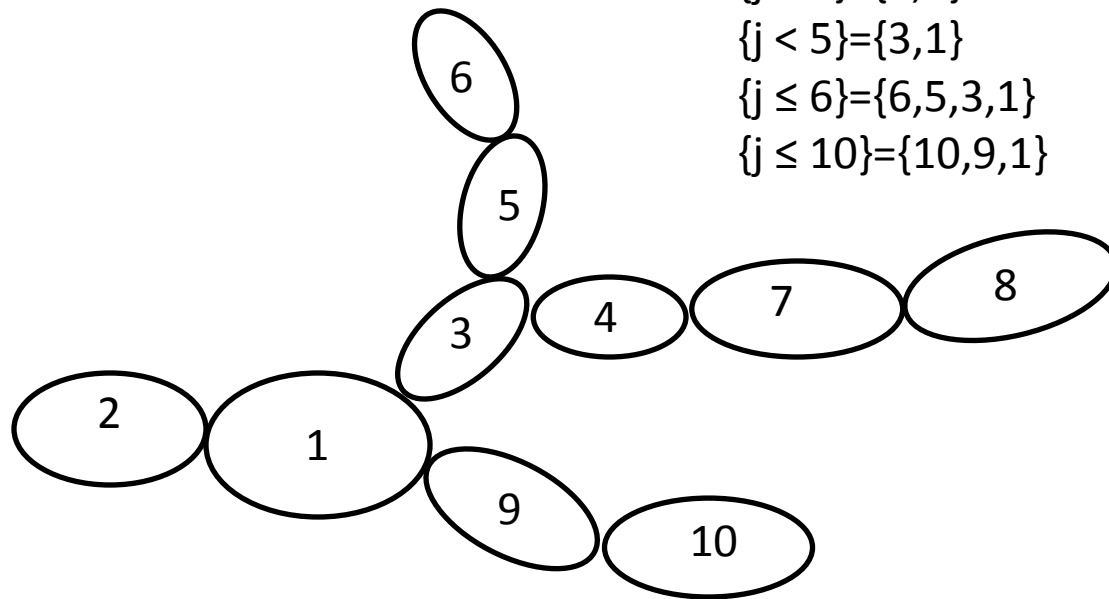
where  $m =$  nilpotency of  $\Phi_A$

- It follows that  $\Phi_A^T$  is square and strictly upper triangular, and  $\Psi_A^T = (e - \Phi_A^T)^{-1}$  exists and can be expressed as

$$\Psi_A^T = e + \Phi_A^T + \Phi_A^{T,2} + \dots + \Phi_A^{T,m-1}$$

- The power series expansion of  $\Psi_A$  and  $\Psi_A^T$  serve to show the existence of  $(e + \Phi_A)^{-1}$  and  $(e + \Phi_A^T)^{-1}$ . More efficient way of computing  $\Psi_A x$  and  $\Psi_A^T x$  for any  $x \in \mathbb{R}^{6N}$  is shown shortly.

# Fig. 2 Example Body Sets



10 body example

$$\{j \leq 2\} = \{2, 1\}$$

$$\{j < 5\} = \{3, 1\}$$

$$\{j \leq 6\} = \{6, 5, 3, 1\}$$

$$\{j \leq 10\} = \{10, 9, 1\}$$

$$\text{branch}_1 = \{j \geq 1\} = \{1:10\}$$

$$\text{branch}_2 = \{j \geq 2\} = \{2\}$$

$$\text{branch}_3 = \{j \geq 3\} = \{3:8\}$$

$$\text{branch}_4 = \{j \geq 4\} = \{4, 7, 8\}$$

$$\bar{1} = \{2, 3, 9\}$$

$$\bar{2} = \bar{6} = \bar{8} = \bar{10} = \{\text{empty}\}$$

$$\bar{3} = \{4, 5\}$$

$$\bar{4} = \{7\}$$



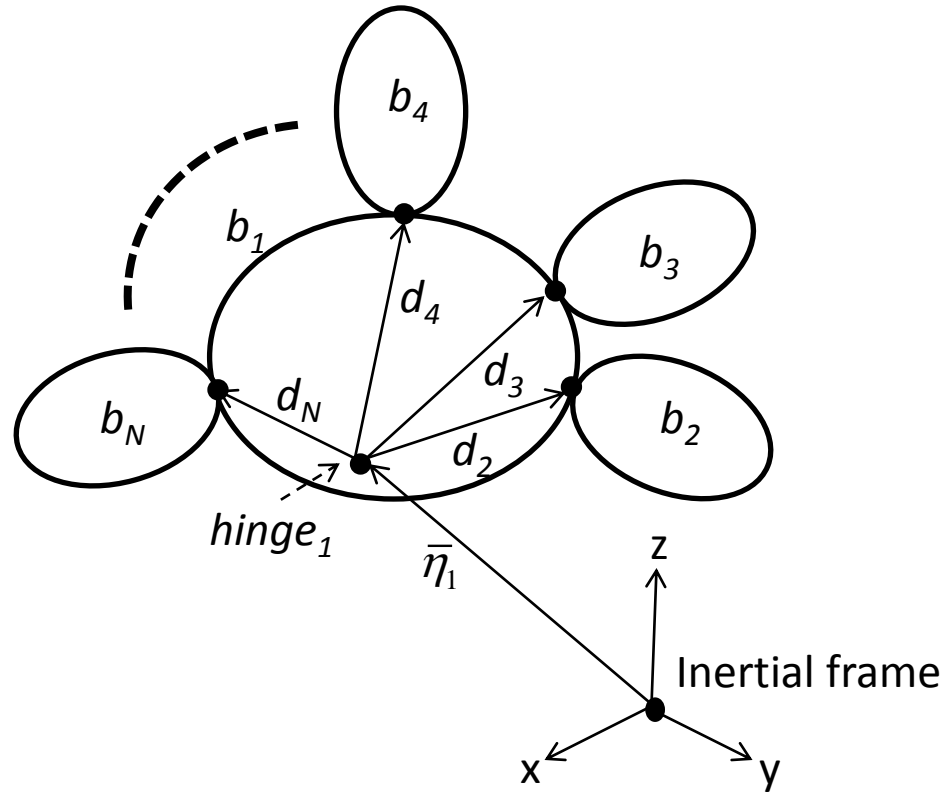
# $\Psi_A$ For Fig. 2 Example

If  $A_i$  is a square submatrix  $\forall i$  for this example, then

$$\Psi_A = (e - \Phi_A)^{-1} = \begin{bmatrix} e & & & & & & & & & \\ A_2 & e & & & & & & & & \\ A_3 & 0 & e & & & & & & & \\ A_4 A_3 & 0 & A_4 & e & & & & & & \\ A_5 A_3 & 0 & A_5 & 0 & e & & & & & \\ A_6 A_5 A_3 & 0 & A_6 A_5 & 0 & A_6 & e & & & & \\ A_7 A_4 A_3 & 0 & A_7 A_4 & A_7 & 0 & 0 & e & & & \\ A_8 A_7 A_4 A_3 & 0 & A_8 A_7 A_4 & A_8 A_7 & 0 & 0 & A_8 & e & & \\ A_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & \\ A_{10} A_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10} & e \end{bmatrix}$$

$\text{nilpotency}(\Phi_A) = 5$

# Fig. 3 Nilpotency 2 System



- By Defs. 2 and 3, if  $A$  is a square matrix, then  $\Psi_A$  for nilpotency 2 systems are

$$\Psi_A = e + \Phi_A$$

with

$$\Phi_A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_2 & 0 & 0 & \cdots & 0 \\ A_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- A nilpotency 2 tree configured system has all bodies  $b_{2:N}$  as immediate children of  $b_1$ . See Fig. 3. For these systems

$$\Psi_A = \begin{bmatrix} e & 0 & 0 & \cdots & 0 \\ A_2 & e & 0 & \cdots & 0 \\ A_3 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & 0 & 0 & \cdots & e \end{bmatrix}$$



# $O(N)$ Recursive Algorithms

- Example presented earlier showed the nature of  $\Psi_A$  given the parent-child relation of bodies in the system.
- Computing expressions involving  $(\Psi_A x)$  or  $(\Psi_A^T x)$  by direct matrix-vector multiplication is not efficient.
- The next two algorithms are order( $N$ ) computation of those expressions.

# Algorithm for $z = \Psi_A x$

- A stacked vector given by  $z = \Psi_A x$  is the result of solving  $z = \Phi_A z + x$  for  $z$ , where  $\Phi_A$  is square and strictly lower triangular per Def 1. Thus, given  $x$ , the elements of  $z$  are obtained by the algorithm below :
  1. set  $z_1 = x_1$  since the first row of  $\Phi_A$  is zero
  2. for  $i = 2 : N$ 
$$z_i = A_i z_{ip} + x_i \quad ;$$
end
- $\Psi_A$  is an order( $N$ ) algorithm.

# Algorithm for $z = \Psi_A^T x$

- A stacked vector given by  $z = \Psi_A^T x$  is the result of solving  $z = \Phi_A^T z + x$  for  $z$ , where  $\Phi_A^T$  is square and strictly upper triangular. Thus, given  $x$ , the elements of  $z$  are obtained by the algorithm below :
  1. set  $z = x$  since the first column of  $\Phi_A^T$  is zero
  2. for  $i = N : 2$ 
$$z_{ip} := z_{ip} + A_i^T z_i \quad ;$$
end
- $\Psi_A^T$  is an order( $N$ ) algorithm.

# Acceleration Equations

- Relation between  $\bar{v}_j$  and  $v_j$  is (See Fig. 4 )

$$\bar{v}_i = S_i v_i \quad (1)$$

where  $\bar{v}_i = [\omega_i, \dot{r}_i]$ ,  $v_i = [\omega_i, \dot{\eta}_i]$  and  $\omega_i$  = total angular rate of  $b_i$

$\dot{r}_i$  = velocity of  $cm_i$ ,  $\dot{\eta}_i$  = velocity of joint <sub>$i$</sub>

$\dot{\eta}_1 = 0$  since joint<sub>1</sub> is attached to ground in the considered system

$$S_i = \begin{bmatrix} e & 0 \\ -\tilde{s}_i & e \end{bmatrix}, s_i = \text{displacement from joint}_i \text{ to } cm_i$$

- Relation between the two accelerations is

$$\dot{\bar{v}}_i = S_i \dot{v}_i + \bar{a}_i \quad (2)$$

where  $\bar{a}_i = \begin{bmatrix} 0 \\ \omega_i \times (\omega_i \times s_i) \end{bmatrix}$

- Total accelerations of (2) in stacked column form is

$$\dot{\hat{v}} = S\dot{v} + \bar{a} \quad (3)$$

where  $\dot{\hat{v}} = \text{col}[\dot{\hat{v}}_i]_{i=1:N}$ ,  $S = \text{diag}[S_i]_{i=1:N}$ ,  $\dot{v} = \text{col}[\dot{v}_i]_{i=1:N}$ ,  $\bar{a} = \text{diag}[\bar{a}_i]_{i=1:N}$

- Relation between  $v_i$ ,  $v_{ip}$  and the relative rate  $\dot{q}_i$  is

$$v_i = D_i v_{ip} + G_i \dot{q}_i \quad (4)$$

where  $D_i = \begin{bmatrix} e & 0 \\ -\tilde{d}_i & e \end{bmatrix}$ ,  $D_1 = 0$ ,  $v_{1p} = 0$

$G_i = [g_i, 0_{3 \times 1}]$  for  $i = 1:N$

- Given (4), the relation between the total acceleration  $\dot{v}_i$ ,  $\dot{v}_{ip}$  and the relative acceleration  $\ddot{q}_i$  is

$$\dot{v}_i = D_i \dot{v}_{ip} + G_i \ddot{q}_i + a_i \quad (5)$$

$$\text{where } a_i = \dot{D}_i v_{ip} + \dot{G}_i \dot{q}_i = \begin{bmatrix} \omega_{ip} \times g_i \dot{\theta}_i \\ \omega_{ip} \times (\omega_{ip} \times d_i) \end{bmatrix}, \quad a_1 = 0 \quad (5a)$$

- Given Def. 1 and Prop. 1, Eq. (5) can be put in column vector form as

$$\dot{v} = \Phi_D \dot{v} + G \ddot{q} + a \quad (6)$$

where,  $\dot{v} = \text{col}[\dot{v}_i]_{i=1:N}$ ,  $D = \text{diag}[D_i]_{i=1:N}$ ,  $G = \text{diag}[G_i]_{i=1:N}$ ,  $\ddot{q} = \text{col}[\ddot{q}_i]_{i=1:N}$

$$a = \text{col}[a_i]_{i=1:N}$$

- Total acceleration in (6) can be solved to be

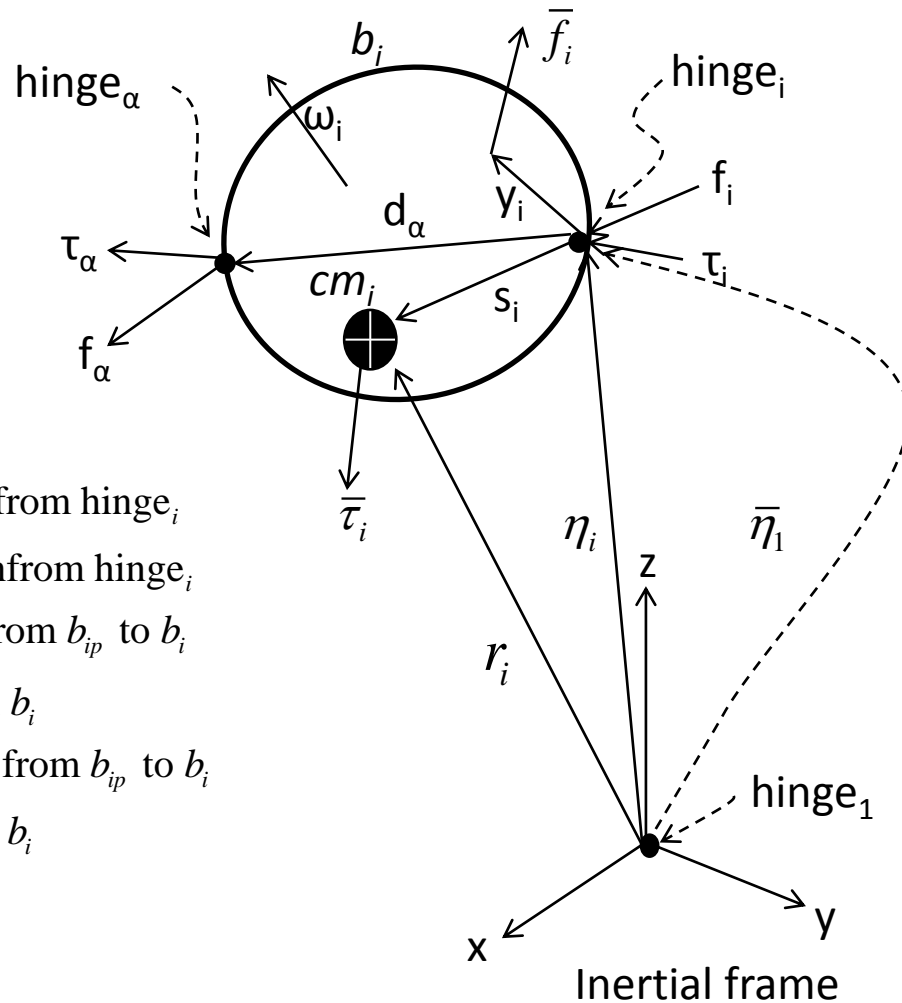
$$\dot{v} = \Psi_D (G \ddot{q} + a) \quad (7)$$

where  $\Psi_D = (e - \Phi_D)^{-1}$

- Given (7), Equation (3) becomes

$$\dot{\bar{v}} = S\Psi_D(G\ddot{q} + a) + \bar{a} \quad (8)$$

# Fig. 4 Forces and Torque on $b_i$



- $b_i$  body  $i$
- $d_\alpha$  hinge $_\alpha$  position from hinge $_i$
- $y_i$   $\bar{f}_i$  impact position from hinge $_i$
- $f_i$  interbody force from  $b_{ip}$  to  $b_i$
- $\bar{f}_i$  external force on  $b_i$
- $\tau_i$  interbody torque from  $b_{ip}$  to  $b_i$
- $\bar{\tau}_i$  external force on  $b_i$
- $\omega_i$  angular rate of  $b_i$



# Dynamics Equations

- Newton-Euler equations for each body  $b_i$  of an *MBS* per Fig. 4 are

$$\begin{aligned} \tau_i - s_i \times f_i = I_i \dot{\omega}_i + \omega_i \times I_i \omega_i - \bar{\tau}_i - (y_i - s_i) \times \bar{f}_i \\ + \sum_{\alpha \in \bar{i}} (\tau_\alpha + (d_\alpha - s_j) \times f_\alpha) \end{aligned} \quad (9)$$

$$f_i = m_i \ddot{x}_i - \bar{f}_i + \sum_{\alpha \in \bar{i}} f_\alpha \quad (10)$$

- (9) and (10) can be made into a 6 x 1 equation to become

$$F_i - \sum_{\alpha \in \bar{i}} D_\alpha^T F_\alpha = S_i^T \bar{M}_i \dot{\bar{v}}_i + \bar{b}_i - Y_i^T \bar{F}_i \quad (11)$$

where  $F_i = \begin{bmatrix} \tau_i \\ f_i \end{bmatrix}$ ,  $D_\alpha^T = \begin{bmatrix} e & \tilde{d}_\alpha \\ 0 & e \end{bmatrix}$ ,  $\bar{M}_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i e \end{bmatrix}$ ,  $\dot{\bar{v}}_i = \begin{bmatrix} \dot{\omega}_i \\ \ddot{r}_i \end{bmatrix}$ ,  $\bar{b}_i = \begin{bmatrix} \omega_i \times I_i \omega_i \\ 0 \end{bmatrix}$

$$Y_i^T = \begin{bmatrix} e & \tilde{y}_i \\ 0 & e \end{bmatrix}, \bar{F}_i = \begin{bmatrix} \bar{\tau}_i \\ \bar{f}_i \end{bmatrix} \text{ (see Fig. 4 for def. of variables.)}$$

$$F_i = \begin{bmatrix} \tau_i \\ f_i \end{bmatrix}, \text{ joint torque-force pair}$$

$$D_\alpha^T = \begin{bmatrix} e & \tilde{d}_\alpha \\ 0 & e \end{bmatrix}, \text{ parent-to-child influence matrix}$$

$$\bar{M}_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i e \end{bmatrix}, \text{ cm centered mass matrix}$$

$$\dot{\bar{v}}_i = \begin{bmatrix} \dot{\omega}_i \\ \ddot{r}_i \end{bmatrix}, \text{ acceleration pair}$$

$$\bar{b}_i = \begin{bmatrix} \omega_i \times I_i \omega_i \\ 0 \end{bmatrix}, \text{ nonlinear force pair}$$

$$Y_j^T = \begin{bmatrix} e & \tilde{y}_j \\ 0 & e \end{bmatrix}, \text{ backward moment-force influence matrix}$$

$$\bar{F}_j = \begin{bmatrix} \bar{\tau}_j \\ \bar{f}_j \end{bmatrix}, \text{ external torque-force pair}$$

- $f_i = G_i^T F_i$  is the actuation force/torque from  $b_{ip}$  to  $b_i$  along the moving joint axis. The complement of that force in  $F_i$  are the constraint forces/torque that prevent motion along the constrained axes of joint( $i$ ).

- Given Def. 2 and Prop. 2, (11) can be put in column vector form as

$$F - \Phi_D^T F = S^T \bar{M} \dot{\bar{v}} + \bar{b} - Y^T \bar{F} \quad (12)$$

where  $F = \text{col}[F_i]_{i=1:N}$ ,  $D = \text{diag}[D_i]_{i=1:N}$ ,  $S^T = \text{diag}[S_i^T]_{i=1:N}$ ,  $\bar{M} = \text{diag}[\bar{M}_i]_{i=1:N}$ ,

$$\dot{\bar{v}} = \text{col}[\dot{\bar{v}}_i]_{i=1:N}, \bar{b} = \text{col}[\bar{b}_i]_{i=1:N}, Y^T = \text{diag}[Y_i^T]_{i=1:N}, \bar{F} = \text{col}[\bar{F}_i]_{i=1:N}$$

- $F$  in (12) is solved to be

$$F = \Psi_D^T (S^T \bar{M} \dot{\bar{v}} + \bar{b} - Y^T \bar{F}) \quad (13)$$

where  $\Psi_D^T = (e - \Phi_D^T)^{-1}$

- Use (8) in (13) to get

$$F = \Psi_D^T (S^T \bar{M} (S \Psi_D (G \ddot{q} + a) + \bar{a}) + \bar{b} - Y^T \bar{F}) \quad (14)$$

- After some rearrangement, (14) becomes

$$F = \Psi_D^T M \Psi_D G \ddot{q} + \Psi_D^T (M \Psi_D a + B - Y^T \bar{F}) \quad (15)$$

where  $M = S^T \bar{M} S = \text{diag}\{S_i^T \bar{M}_i S_i\} = \text{diag}\{M_i\}_{i=1:N}$  (16)

$$M_i = \begin{bmatrix} I_i - m_i \tilde{s}_i \tilde{s}_i & m_i \tilde{s}_i \\ -m_i \tilde{s}_i & m_i e \end{bmatrix}, \text{ joint centered mass matrix}$$

$$B = S^T \bar{M} \bar{a} + \bar{b} = \text{col}\{B_i\}_{i=1:N} \quad (17)$$

$$B_i = \begin{bmatrix} \omega_i \times (I_i - m_i \tilde{s}_i \tilde{s}_i) \omega_i \\ m_i \omega_i \times (\omega_i \times s_i) \end{bmatrix}, \text{ nonlinear force}$$

- By projecting  $F$  of (15) onto the free joint axes, the dynamics equations becomes

$$\mathcal{M}\ddot{q} = f - C(\theta, \dot{\theta}) \quad (18)$$

where  $f = G^T F$ , actuation force/torque along free axes (18a)

$$C(\theta, \dot{\theta}) = G^T \Psi_D^T (M \Psi_D \dot{a} + B - Y^T \bar{F}) \quad (18b)$$

$$\mathcal{M} = G^T \Psi_D^T M \Psi_D G, \text{ system mass matrix} \quad (18c)$$

- Acceleration state vector in (18) can be solved to be

$$\ddot{q} = \mathcal{M}^{-1}(f - C(\theta, \dot{\theta})) \quad (19a)$$

$$= [e - K^T \Phi_D \Psi_\beta G] R^{-1} [e - G^T \Psi_\beta^T \Phi_D^T K] (f - C(\theta, \dot{\theta})) \quad (19b)$$

(See [1][2] on the factorization of  $\mathcal{M}^{-1}$ . Note that the  $\mathcal{E}$  operator in [2] is the  $\Phi$  operator here. )

where  $\ddot{q} = [\ddot{\theta}_i]_{i=1:N}$

$$\beta = \text{diag}[(e - G_i K_i^T) D_i]_{i=1:N}$$

$$D = \text{diag}[D_i]_{i=1:N}, \text{ with } D_1 = 0_{6 \times 6}$$

$$R^{-1} = \text{diag}[R_i^{-1}]_{i=1:N}, R_i = G_i^T J_i G_i$$

$$J_i = \text{branch}_i \text{ inertia tensor}$$

$$K = \text{diag}[K_i]_{i=1:N}, K_i = J_i G_i R_i^{-1}$$

$$G = \text{diag}[G_i]_{i=1:N}$$

- Backward recursive computation of branch inertia tensors  $\{J_i\}_{i=1:N}$  :
  - a) for  $i = 1:N$ ;  $J_i = M_i$ ; end
  - b) for  $i = N:2$ 

$$\hat{J}_i = J_i - J_i G_i R_i^{-1} G_i^T J_i$$

$$J_{ip} = J_{ip} + D_i^T \hat{J}_i D_i$$
 end

(see Ref. [1] and [2].)

- Note that (18a) follows from the definition of  $f$ . Each  $f_i$  of  $f$  is the net force/torque from the control and the spring/damper forces transmitted along the free axis of joint $_i$ .
- Eqs. (19a) and (19b)[2] are two acceleration state solution options.
- Eq. (19a) requires the assembly of  $\mathcal{M}$  using (18c). While that matrix is factored is an advantage in the assembly process. However, acceleration solution of (18) by any linear equation solvers involving  $\mathcal{M}$  is an order( $N^3$ ) process.
- Eq. (19b) requires that  $(f - C(\theta, \dot{\theta}))$  be operated from right to left using the indicated operators. Every one of those serial operations is an  $O(N)$  process. Therefore, the solution process by (19b) is  $O(N)$ .



# Forward & Inverse Dynamics

- Forward dynamics regarding (18) is the problem of solving for the system accelerations  $\ddot{q}$  given the interbody forces along the free joint axes and the external forces, i.e.  $\{f_i, \bar{F}_i\}$ . This is effectively (19a, or 19b).
- Inverse dynamics regarding (18) is the calculation of the required interbody forces along the joint free axes  $\{f_i\}$  given the system accelerations and the external forces, i.e.  $\{\ddot{q}_i^*, \bar{F}_i\}$ .

# Constraint Forces

- Given that the system accelerations are solved by (19a, or 19b), then all the interbody forces  $F_i$  can be determined by (15).
- Proposition 5. Given that  $f_i = G_i^T F_i$  is the actuation force/torque along the free axis of  $b_i$ 's inboard hinge, then  $f_i^c = \hat{G}_i^c F_i$  is the constraint forces/torque at the same hinge, where

$$\hat{G}_i \in \mathbb{R}^{6 \times 5}, \text{ constraint matrix; } G_i^T \hat{G}_i = 0_{1 \times 5}, \quad G_i^T G_i = 1, \quad \hat{G}_i^T \hat{G}_i = e_{5 \times 5} \quad (20)$$

More specifically, if  $\bar{G}_i^c$  is a particular column of  $\hat{G}_i$ , then  $\bar{f}_i^c = \bar{G}_i^{cT} F_i$  is the constraint torque (or force) along  $\bar{G}_i^c$ .

- Example: Let  $x$  - *axis* be the free axis of joint<sub>*i*</sub> frame and the request is for the constraint force along the  $z$  - *axis* of that frame. Given that  $F_i$  has been computed by (15), then

$$G_i = \begin{bmatrix} e_i^x, 0_{3 \times 1} \end{bmatrix}, \quad \bar{G}_i^c = \begin{bmatrix} 0_{3 \times 1}, e_i^z \end{bmatrix}$$

Ans:  $\bar{f}_i^c = \begin{bmatrix} 0_{3 \times 1}, e_i^z \end{bmatrix}^T F_i$

where  $e_i^z =$  unit  $z$  - *axis* of joint<sub>*i*</sub> frame in inertial coordinates.

# Computing System Mass Matrix

Procedure to compute  $\mathcal{M}$  per (18c):

1. Compute  $J = \Psi_D G$   $\Rightarrow$  2. Compute  $Q = MJ$   
with  $J = [J(:,1) J(:,2) \cdots J(:,N)]$  with  $Q = [Q(:,1) Q(:,2) \cdots Q(:,N)]$   
 $G = [G(:,1) G(:,2) \cdots G(:,N)]$   $M = \text{diag}\{M(i), i = 1:N\}$   
for  $i = 1:N$  for  $i = 1:N$   
 $J(:,i) = \Psi_D G(:,i)$  ; per Alg.1  $Q(i,:) = M(i)J(i,:)$  ; row block matrix mult.  
end end
3.  $\mathcal{M} = J^T Q$  , end

Q1: what is the order of the above procedure?

Q2: can it be made order( $N$ )?

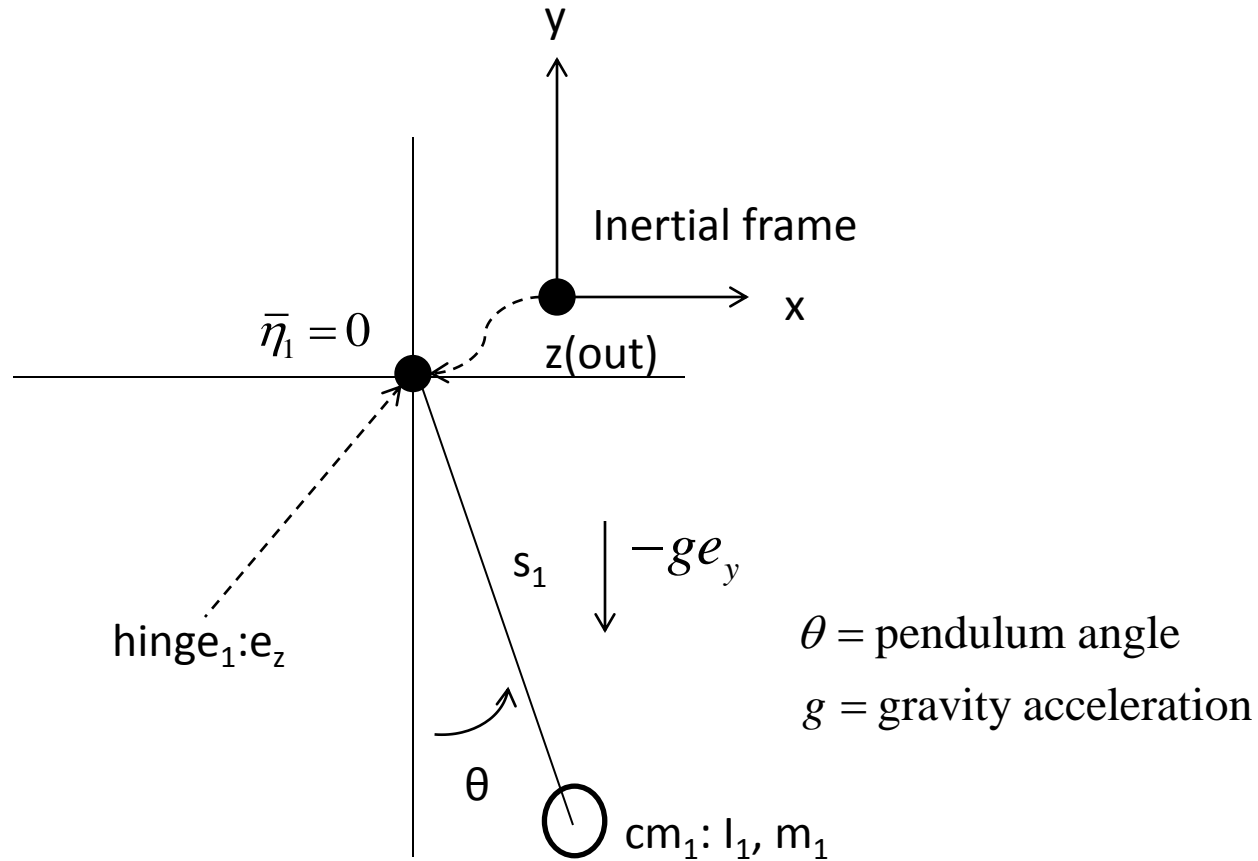
- Ans1: Steps 1, 2 and 3 are each order( $N^2$ ), therefore the process is order( $N^2$ )
- Ans2: No. Computation of  $\mathcal{M}$  is order( $N^2$ ) per Ans1.

# $O(N)$ Solution of EOM

- For  $t = \{t_0, t_1, t_2, \dots, t_{end}\}$ , do {
  - Step 1: Given  $[q, \dot{q}]$  at  $t_k$  compute  $\ddot{q}$  :
    - a. compute  $\{C_i\}$  given  $\{q_i, \theta_i\}_{i=1:N} \in q$
    - b. compute  $\{I_i, g_i, d_i, s_i, r_i, \eta_i, \bar{y}_i, D_i, G_i, M_i, J_i, K_i, R_i, \beta_i\}_{i=1:N}$
    - c. compute  $\{\omega_i, \dot{r}_i, \dot{\eta}_i\}_{i=1:N}$  given #a, #b and  $\dot{q}$
    - d. compute  $\{\bar{\tau}_i, \bar{f}_i, f_i\}_{i=1:N}$ , application specific input
    - e. compute  $\{a, B\}$  per (5a) and (17)
    - f. solve (18) for  $\ddot{q}$ 
      - $O(N^3)$  opt1: compute  $(f - C(\theta, \dot{\theta}) \wedge \mathcal{M})$  per (18a,b,c) and use (19a)
      - $O(N)$  opt2: compute  $(f - C(\theta, \dot{\theta}))$  per (18a,b) and use (19b)
  - Step 2: Given  $[q, \dot{q}, \ddot{q}]$  at  $t_k$ , integrate  $[q, \dot{q}]$  to  $t_{k+1}$  }

# Example 1

## Simple Pendulum



# Model Parameters

$$\bar{\eta}_1 = 0, \omega_1 = \dot{\theta} e_z$$

$$s_1 = L(\sin(\theta)e_x - \cos(\theta)e_y) \quad ; \quad \text{cm position from hinge}$$

$$\bar{F}_1 = \begin{bmatrix} 0 \\ -m_1 g_y \end{bmatrix} \quad ; \quad \text{external force}$$

$$f = \tau = G_1^T F_1 \quad ; \quad \text{hinge torque}$$

$$G = G_1, \text{ with } G_1 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \quad \text{(M1)}$$

$$\Phi_D = 0_{6 \times 6} \quad \text{(M2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e_{6 \times 6} \quad \text{(M3)}$$



- Let  $C(\theta, \dot{\theta}) = G^T \Psi_D^T (M \Psi_D a + B - Y^T \bar{F})$  per (18b) with  $a = 0$  and  $\Psi_D = e$  for this example.

Thus, given (17) and (M3), we have

$$Y^T = \begin{bmatrix} e & \tilde{s}_1 \\ 0 & e \end{bmatrix}$$

$$B = [0, -m_1 \dot{\theta}^2 s_1]$$

$$Y^T \bar{F} = [-\tilde{s}_1 m_1 g e_y, -m_1 g e_y]$$

$$\Psi_D^T (B - Y^T \bar{F}) = [\tilde{s}_1 m_1 g e_y, -m_1 \dot{\theta}^2 s_1 + m_1 g e_y]$$

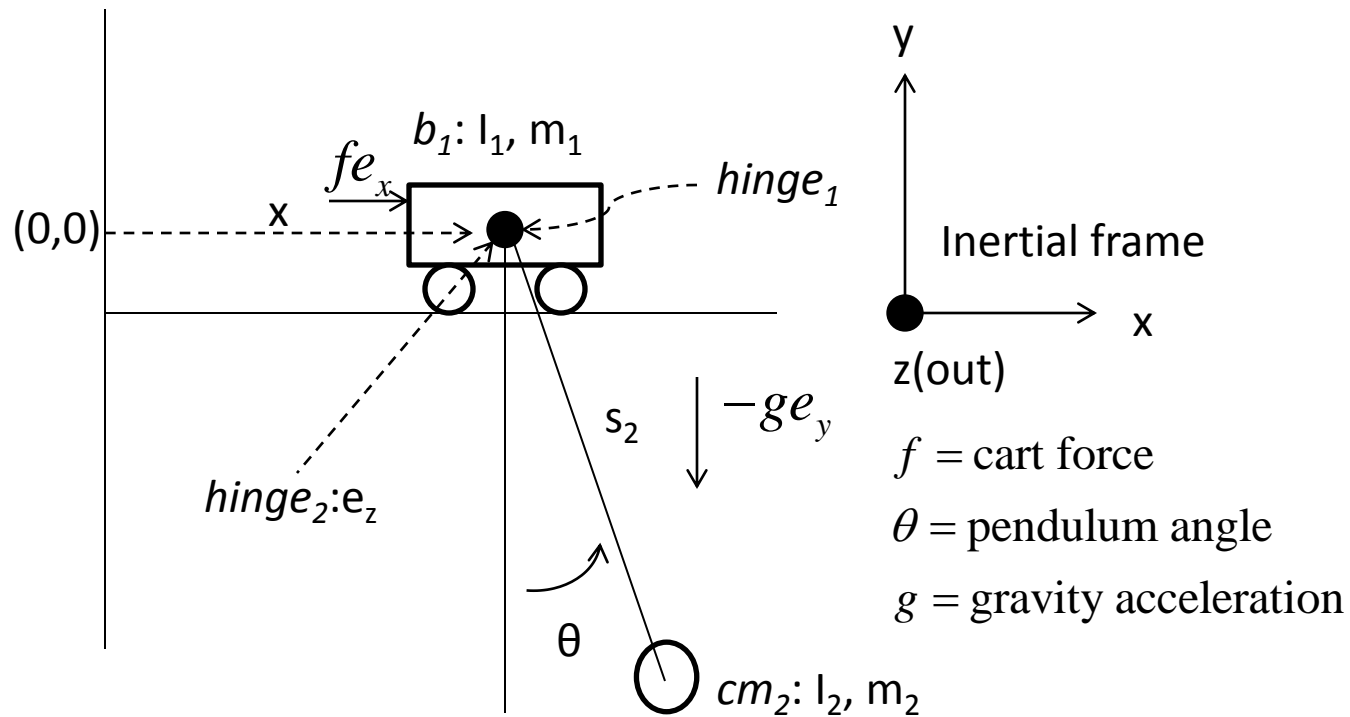
$$f - C(\theta, \dot{\theta}) = \tau - m_1 g L \sin(\theta)$$

- *EOM* per (18) for this pendulum is

$$(I_1^{zz} + m_1 L^2) \ddot{\theta} = \tau - m_1 g L \sin(\theta) \quad (\text{M4})$$

# Example 2

## Pendulum with a Moving Base



# Model Parameters

$$d_2 = 0, \bar{\eta}_1 = xe_x, \omega_1 = 0, \omega_2 = \dot{\theta}e_z$$

$$s_1 = 0, s_2 = L(\sin(\theta)e_x - \cos(\theta)e_y) \quad ; \quad \text{cm position from hinge}$$

$$\bar{F}_1 = \begin{bmatrix} 0 \\ fe_x - m_1ge_y \end{bmatrix}, \bar{F}_2 = \begin{bmatrix} 0 \\ -m_2ge_y \end{bmatrix} \quad ; \quad \text{external forces}$$

$$\text{Let } f = G^T F = [0, 0] \quad ; \quad \text{joint transmitted force/torque}$$

$$G = \text{diag}\{G_1, G_2\}, \text{ with } G_1 = \begin{bmatrix} 0 \\ e_x \end{bmatrix}, G_2 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \quad (\text{N1})$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, \text{ where } 0 = 0_{6 \times 6} \quad \text{nilpotency}(\Phi_D) = 2 \quad (\text{N2})$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6 \times 6} & 0 \\ e_{6 \times 6} & e_{6 \times 6} \end{bmatrix} \quad (\text{N3})$$

- Let  $C(\theta, \dot{\theta}) = G^T \Psi_D^T (M \Psi_D a + B - Y^T \bar{F})$  per (18b), and we have  $a = 0$  and  $\omega_1 = 0$  for this example. Given (17) and (N3), we have

$$B = [[0, 0], [0, -m_2 \dot{\theta}^2 s_2]]$$

$$Y_1^T = 0_{6 \times 6}, Y_2^T = \begin{bmatrix} e & \tilde{s}_2 \\ 0 & e \end{bmatrix}$$

$$Y^T \bar{F} = [[0, f e_x - m_1 g e_y], [-\tilde{s}_2 m_2 g e_y, -m_2 g e_y]]$$

$$\Psi_D^T (B - Y^T \bar{F}) = [[\tilde{s}_2 m_2 g e_y, -f e_x + m_1 g e_y - m_2 \dot{\theta}^2 s_2 + m_2 g e_y], \\ [\tilde{s}_2 m_2 g e_y, m_2 g e_y - m_2 \dot{\theta}^2 s_2]]$$

$$C(\theta, \dot{\theta}) = [-f - m_2 L s(\theta) \dot{\theta}^2, m_2 g L s(\theta)]$$

Given (18a) and  $f = [0, 0]$  for this case,

$$f - C(\theta, \dot{\theta}) = [f + m_2 L s(\theta) \dot{\theta}^2, -m_2 g L s(\theta)]$$

- EOM for this pendulum on a moving base is

$$\begin{bmatrix} m_1 + m_2 & m_2 L \cos(\theta) \\ m_2 L \cos(\theta) & I_2^{zz} + m_2 L^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f + m_2 L \sin(\theta) \dot{\theta}^2 \\ -m_2 g L \sin(\theta) \end{bmatrix} \quad (\text{N4})$$

The system mass matrix in (N4) was derived using Eq. (18c)

- Compute  $\mathcal{M}$  per (18c) for this example by the following procedure
  1. Compute  $J = \Psi_D G$  per matrix column version of Alg. 1, with  $G = \text{diag}\{G_i\}_{i=1:2}$
  2. Compute  $Q = MJ$  where  $M = \text{diag}\{M_i\}_{i=1:2}$
  3. Compute  $\mathcal{M} = G^T \Psi_D^T M \Psi_D G = J^T Q$ , with  $\mathcal{M} \in R^{2 \times 2}$

# Summary

- *EOM* for a tree configured multibody system using generalized coordinates and Newton-Euler equations has been presented.
- *EOM* formulation here requires no joint constraints to enforce. One can extract the interbody constraint force or torque at any joint from Eq. (15), once the system accelerations have been determined.
- $O(N^3)$  and  $O(N)$  solution to the *EOM*, Eq. (18), have been presented. (See Eq. (19a) and Eq. (19b).)

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