Kinematics of MBS

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Introduction

- Kinematics of a mechanism is about computing the attitudes, positions, velocities and accelerations of member bodies of the mechanism in some reference frame given the generalized coordinates, rates and accelerations of the mechanism.

- Minimal set of coordinates that make these computations possible are called the generalized coordinates of the mechanism or mechanical system.

- **Kinematics of rigid multibody systems that have only single axis rotational motion between joint connected bodies is considered here.**

- A large group of mechanisms is covered even with this restriction. This group includes robots arms, ground fixed gimbaled antennas, multi-link pendulums.

- The generalized coordinates for this group of systems is

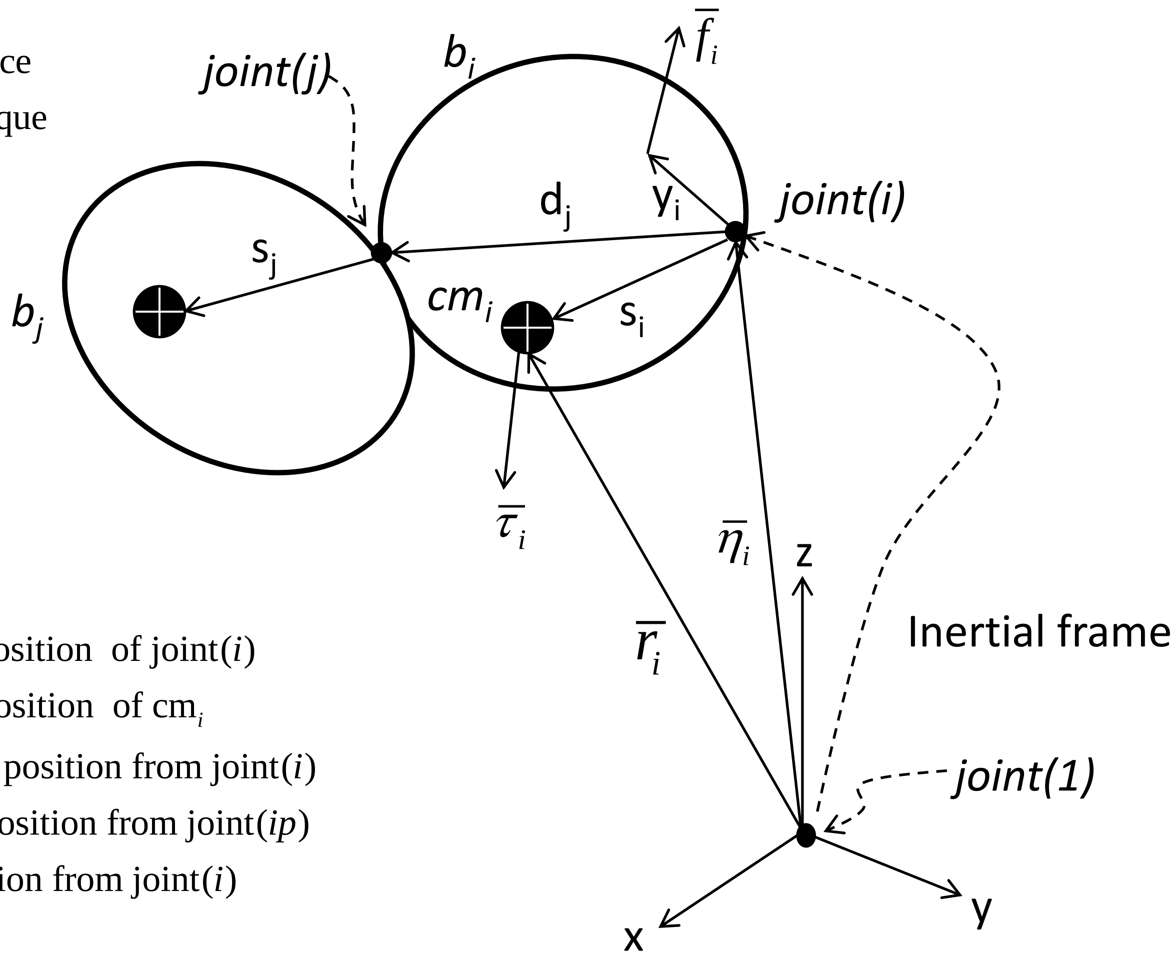
$$q = \{\theta_i\}_{i=1:N}$$

where $\theta_i =$ inboard joint angle of b_i

Fig. 1 Notations

$\overline{f}_i$ external force

$\overline{\tau}_i$ external torque



b_i body i

$\overline{\eta}_i$ inertial position of joint(i)

$\overline{r}_i$ inertial position of cm_i

y_i $\overline{f}_i$ impact position from joint(i)

d_i joint(i) position from joint(ip)

s_i cm_i position from joint(i)

- The root body b_1 is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. Generally, the choice of b_1 is arbitrary. In the case of a humanoid robot, b_1 could be the head, the torso or the hip.

- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.

- The chain of bodies between b_1 and b_j shall be denoted as $\{i \mid i \leq j\}$ or just $i \leq j$. The less-than-or-equal relation in this case means a topological order not a numerical order.

- The set of bodies branching from b_j shall be denoted as $\{i \mid i \geq j\}$ or just $i \geq j$. The greater-than-or-equal relation here means a topological order not a numerical order.

- All vectors in the following discussion are given in the format x_j^i . The subsript j denotes the body that x belongs to and the superscript i denotes the coordinate frame that the vector is in.

- Vectors with no superscript are given in inertial coordinates unless defined otherwise

Coordinate Frames

- A Cartesian reference frame is a 3 dimensional space with its origin defined at some point of interest. The translational motion of objects are defined by the [x, y, z] coordinates of the objects over time in that frame. The attitude motion of each body is defined by a direction cosine matrix that maps vectors fixed on that body to the reference frame over time.
- A local reference frame shall mean a body fixed reference frame whose origin is located at the inboard hinge point or joint of the body.
- An inertial frame is any frame in which Newtonian dynamics hold.
- Workspace frame is an inertial reference frame chosen to define the motion of a mechanism or an object such as a robot in that frame.

Attitude Equations

- A local reference frame is a body fixed three dimensional coordinate frame with its origin at the inboard joint of the body. For b_1, the local reference frame origin is defined at an arbitrary reference point on it, called *joint(1)* here. See Fig. 1.

- Given $\{\theta_i\}_{i=1:N}$, the inertial direction cosine matrix of every body in a hinge connected system can be computed recursively as

$$C_1 = \bar{C}_1^0[(1-\cos(\theta_1))g_1^1 g_1^{1,T} + \cos(\theta_1)e + \sin(\theta_1)\tilde{g}_1^1] \qquad \text{Eq. 1}$$

for $i = 2 : N$

$$ip = parent(i)$$

$$C_i = C_{ip}\bar{C}_i^{ip}[(1-\cos(\theta_i))g_i^i g_i^{i,T} + \cos(\theta_i)e + \sin(\theta_i)\tilde{g}_i^i] \qquad \text{Eq. 2}$$

end

where $C_i =$ dcm from b_i to inertial frame

$C_i^{ip} =$ dcm from b_i to b_{ip} frame, ip=0 means inertial frame

$\bar{C}_i^{ip} = C_i^{ip}(\theta_i = 0)$

$g_i^i =$ joint(i) rotation axis in b_i coordinates

$\tilde{g}_i^i =$ skew symmetric matrix of g_i^i

- Note that Rodrigues' rotation formula are used in Eqs. (1) and (2) for simplicity

Position Equations

- Given $\{C_i\}_{i=1:N}$, all body fixed vectors in the system can be expressed in inertial coordinates

- *Joint* and *cm* position of every body can be computed in the general reference frame recursively per Fig. 1 as

$$\bar{\eta}_i = \bar{\eta}_{ip} + d_i \quad \text{, for } i = 2:N, \text{ given } \bar{\eta}_1 \qquad \text{Eq. 3}$$

$$\bar{r}_i = \bar{\eta}_i + s_i \quad \text{, for } i = 1:N \qquad \text{Eq. 4}$$

where $d_i =$ displacement from $\bar{\eta}_{ip}$ to $\bar{\eta}_i$

$s_i =$ displacement from $\bar{\eta}_i$ to cm_i

$\bar{\eta}_i =$ inertial position of $hinge_i$

$\bar{r}_i =$ inertial position of cm_i

- By Eqs. 3 and 4, the *joint(i)* position and cm_i position can be written explicitly as

$$\bar{\eta}_i = \bar{\eta}_1 + \sum_{\alpha \le i} d_\alpha \qquad \text{Eq. 5}$$

$$\bar{r}_i = \bar{\eta}_i + s_i \qquad \text{Eq. 6}$$

- Likewise, $\bar{r}_i$ can also be computed recursively per Fig. 1 given $\bar{r}_1$

$$\bar{r}_i = \bar{r}_{ip} - s_{ip} + (d_i + s_i) \quad \text{, for } i = 2:N \qquad \text{Eq. 7}$$

Velocity Equations

- The generalized rates are

$$\dot{q} = \left\{\dot{\theta}_i\right\}_{i=1:N}$$

- Total angular rates can be computed recursively as

$$\omega_i = \omega_{ip} + g_i\dot{\theta}_i \text{ , for } i = 2:N \qquad \text{Eq. 8}$$

where $g_i =$ inboard hinge axis of b_i with $\omega_1 = g_1\dot{\theta}_1$

- By Eq. 8, the total angular rate can be written explicitly as

$$\omega_i = \omega_1 + \sum_{\alpha \le i} g_\alpha \dot{\theta}_\alpha \qquad \text{Eq. 9}$$

- *joint(i)* and cm_i velocities can be computed given $\dot{\bar{\eta}}_1$.

$$\dot{\bar{\eta}}_i = \dot{\bar{\eta}}_{ip} + \omega_{ip} \times d_i \text{ , for } i = 2:N \qquad \text{Eq. 10}$$

$$\dot{\bar{r}}_i = \dot{\bar{\eta}}_i + \omega_i \times s_i \text{ , for } i = 1:N \qquad \text{Eq. 11}$$

- *joint(i)* and cm_i velocities in explicit form are

$$\dot{\bar{\eta}}_i = \dot{\bar{\eta}}_1 + \omega_1 \times (\bar{\eta}_i - \bar{\eta}_1) + \sum_{\alpha < i} g_\alpha \dot{\theta}_\alpha \times (\bar{\eta}_i - \bar{\eta}_\alpha) \qquad \text{Eq. 12}$$

$$\dot{\bar{r}}_i = \dot{\bar{\eta}}_1 + \omega_1 \times (\bar{r}_i - \bar{\eta}_1) + \sum_{\alpha \le i} g_\alpha \dot{\theta}_\alpha \times (\bar{r}_i - \bar{\eta}_\alpha) \qquad \text{Eq. 13}$$

- cm_i velocities can also be computed recursively given $\dot{\bar{r}}_1$

$$\dot{\bar{r}}_i = \dot{\bar{r}}_{ip} + \omega_{ip} \times (r_i - r_{ip}) + \mathrm{g}_i \dot{\theta}_i \times s_i \text{ , for } i = 2:N \qquad \text{Eq. 14}$$

Acceleration Equations

- Angular acceleration of b_i in recursive and explicit forms are

$$\dot{\omega}_i = \dot{\omega}_{ip} + g_i \ddot{\theta}_i + \omega_{ip} \times g_i \dot{\theta}_i \text{ , for } i = 2:N \qquad \text{Eq. 15}$$

$$\dot{\omega}_i = \dot{\omega}_1 + \sum_{\alpha \le i} (g_\alpha \ddot{\theta}_\alpha + \omega_{ip} \times g_i \dot{\theta}_i) \qquad \text{Eq. 16}$$

- Joint(i) and cm_j accelerations in recursive form are

$$\ddot{\bar{\eta}}_i = \ddot{\bar{\eta}}_{ip} + \dot{\omega}_{ip} \times d_i + \omega_{ip} \times (\omega_{ip} \times d_i) \text{ , for } i = 2:N \qquad \text{Eq. 17}$$

$$\ddot{\bar{r}}_i = \ddot{\bar{\eta}}_i + \dot{\omega}_i \times s_i + \omega_i \times (\omega_i \times s_i) \quad \text{ , for } i = 1:N \qquad \text{Eq. 18}$$

- Total accelerations of *joint(i)* and cm_i in explicit form are

$$\ddot{\bar{\eta}}_i = \ddot{\bar{\eta}}_1 + \dot{\omega}_1 \times (\bar{\eta}_i - \bar{\eta}_1) + \omega_1 \times (\dot{\bar{\eta}}_i - \dot{\bar{\eta}}_1) + \sum_{\alpha<i} \left[g_\alpha \ddot{\theta}_\alpha \times (\bar{\eta}_i - \bar{\eta}_\alpha) + (\omega_{\alpha p} \times g_\alpha \dot{\theta}_\alpha) \times (\bar{\eta}_i - \bar{\eta}_\alpha) + g_\alpha \dot{\theta}_\alpha \times (\dot{\bar{\eta}}_i - \dot{\bar{\eta}}_\alpha) \right] \qquad \text{Eq. 19}$$

$$\ddot{\bar{r}}_i = \ddot{\bar{\eta}}_1 + \dot{\omega}_1 \times (\bar{r}_i - \bar{\eta}_1) + \omega_1 \times (\dot{\bar{r}}_i - \dot{\bar{\eta}}_1) + \sum_{\alpha\le i} \left[g_\alpha \ddot{\theta}_\alpha \times (\bar{r}_i - \bar{\eta}_\alpha) + (\omega_{\alpha p} \times g_\alpha \dot{\theta}_\alpha) \times (\bar{r}_i - \bar{\eta}_\alpha) + g_\alpha \dot{\theta}_\alpha \times (\dot{\bar{r}}_i - \dot{\bar{\eta}}_\alpha) \right] \qquad \text{Eq. 20}$$

Other Points of Interest

- Motion of markers on bodies in the system may be needed for performance or constraint evaluations.

- Calculating the k-th position marker on b_i for some i:

$$p_i(k) = C_i p_i^i(k) \qquad \text{Eq. 21}$$

$$\bar{p}_i(k) = \bar{\eta}_i + p_i(k) \qquad \text{Eq. 22}$$

$$\dot{\bar{p}}_i(k) = \dot{\bar{\eta}}_i + \omega_i \times p_i(k) \qquad \text{Eq. 23}$$

$$\ddot{\bar{p}}_i(k) = \ddot{\bar{\eta}}_i + \dot{\omega}_i \times p_i(k) + \omega_i \times (\omega_i \times p_i(k)) \qquad \text{Eq. 24}$$

where $p_i^i(k) =$ marker(k) position on b_i from joint(i) in b_i coordinate

$p_i(k) =$ inertial coordinates of $p_i^i(k)$

$\bar{p}_i(k) =$ inertial position of position marker(k)

$\dot{\bar{p}}_i(k) =$ inertial velocity of position marker(k)

$\ddot{\bar{p}}_i(k) =$ inertial acceleration of position marker(k)

- Motion of unit vectors on member bodies other than the Local Reference Frame axes may be needed for performance or in constraint evaluations.

- Calculating the k-th directional marker on b_i for some i :

$$u_i(k) = C_i u_i^i(k) \qquad \text{Eq. 25}$$

$$\dot{u}_i(k) = \omega_i \times u_i(k) \qquad \text{Eq. 26}$$

$$\ddot{u}_i(k) = \dot{\omega}_i \times u_i(k) + \omega_i \times (\omega_i \times u_i(k)) \qquad \text{Eq. 27}$$

where $u_i^i(k) =$ unit vector of direction marker(k) on b_i in b_i coordinates

$u_i(k) =$ inertial coordinates of $u_i^i(k)$

$\dot{u}_i(k) =$ inertial velocity (turning rate) of $u_i(k)$

$\ddot{u}_i(k) =$ inertial acceleration of $u_i(k)$

Summary

- Kinematics equations in attitude, position, velocity and acceleration in some Cartesian reference frame for a rigid multibody system whose generalized coordinates are $\{\theta_i\}_{i=1:N}$ have been presented.

- Kinematics equations for positional and directional markers in the said reference frame have also been presented. They are needed for performance and constraint evaluations.