

# Joint Motion

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# Introduction

- This presentation addresses the kinematic equations for  $n$ -body systems whose joints can be any of the following types
  - ❖ 1 dof translational  $\iff$  joint type(JT)=1
  - ❖ 3 dof translational  $\iff$  joint type(JT)=2
  - ❖ 1 dof rotational  $\iff$  joint type(JT)=3
  - ❖ 3 dof rotational  $\iff$  joint type(JT)=4
- MBS with Mixed joint types appear often in robotics, and complex vibration isolation mechanisms.

- The generalized coordinates chosen for the considered system is the set:

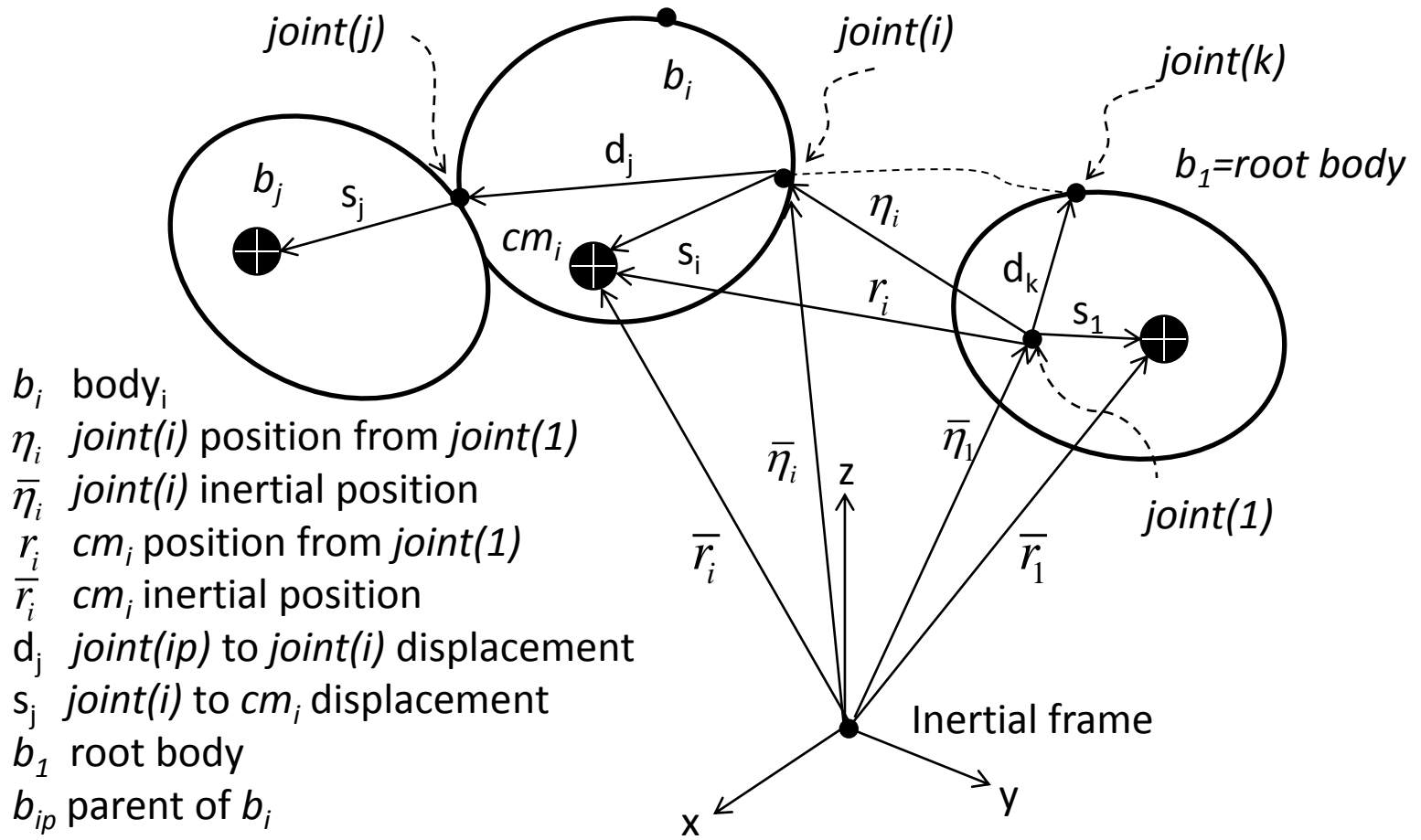
$$q = \{\gamma_i\}_{i=1:N}$$

where

$$\gamma_i = \begin{cases} \delta_i, \text{ scalar displacement;} & \text{JT}(i) = 1 \\ \mathbf{x}_i^i, 3 \times 1 \text{ displacement;} & \text{JT}(i) = 2 \\ \theta_i, \text{ joint angle;} & \text{JT}(i) = 3 \\ \tilde{\varepsilon}_i, \text{ relative attitude quaternion;} & \text{JT}(i) = 4 \end{cases}$$

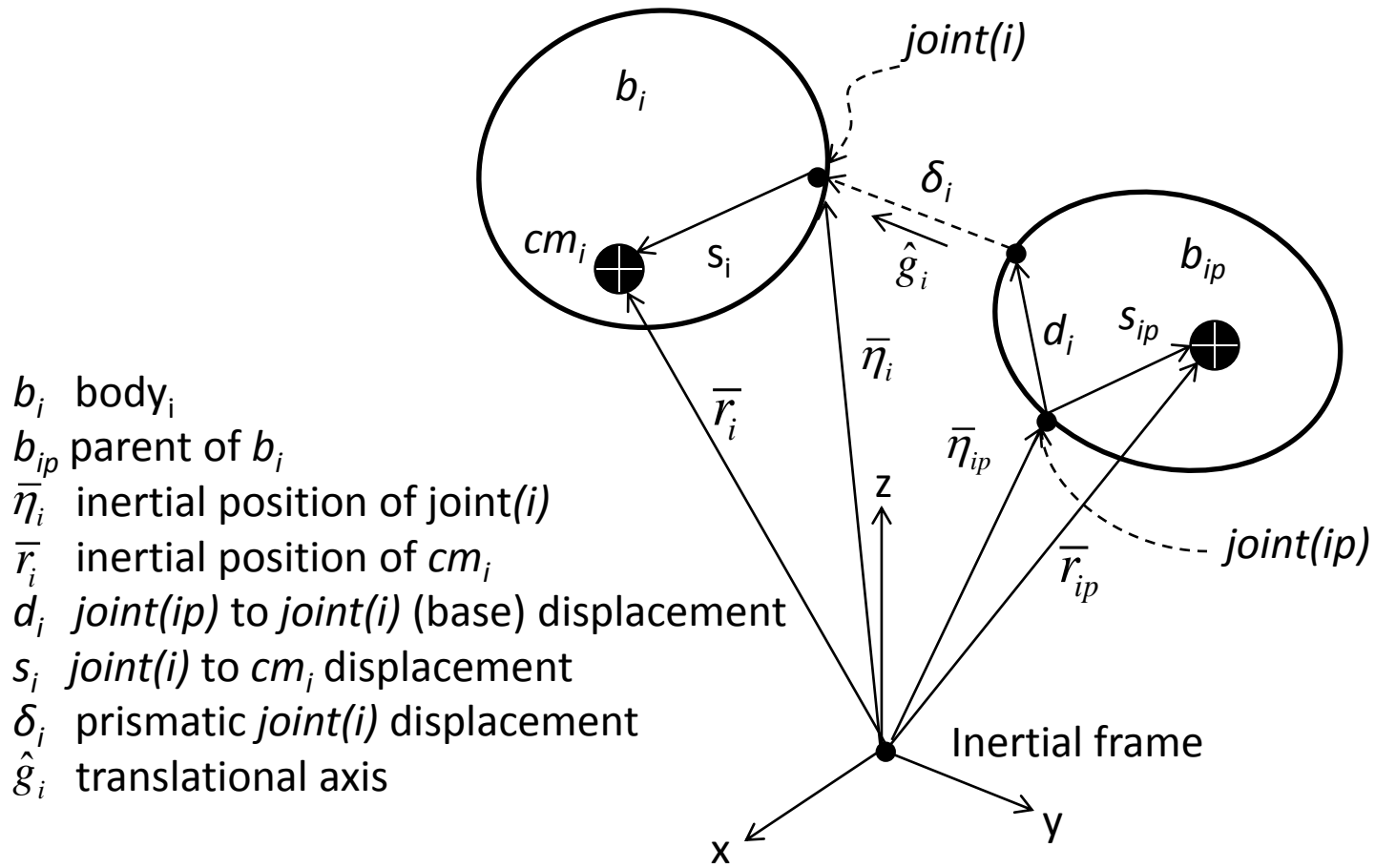
= inboard joint coordinate of  $b_i$

# Fig. 1 Joint 1 Centric Notation

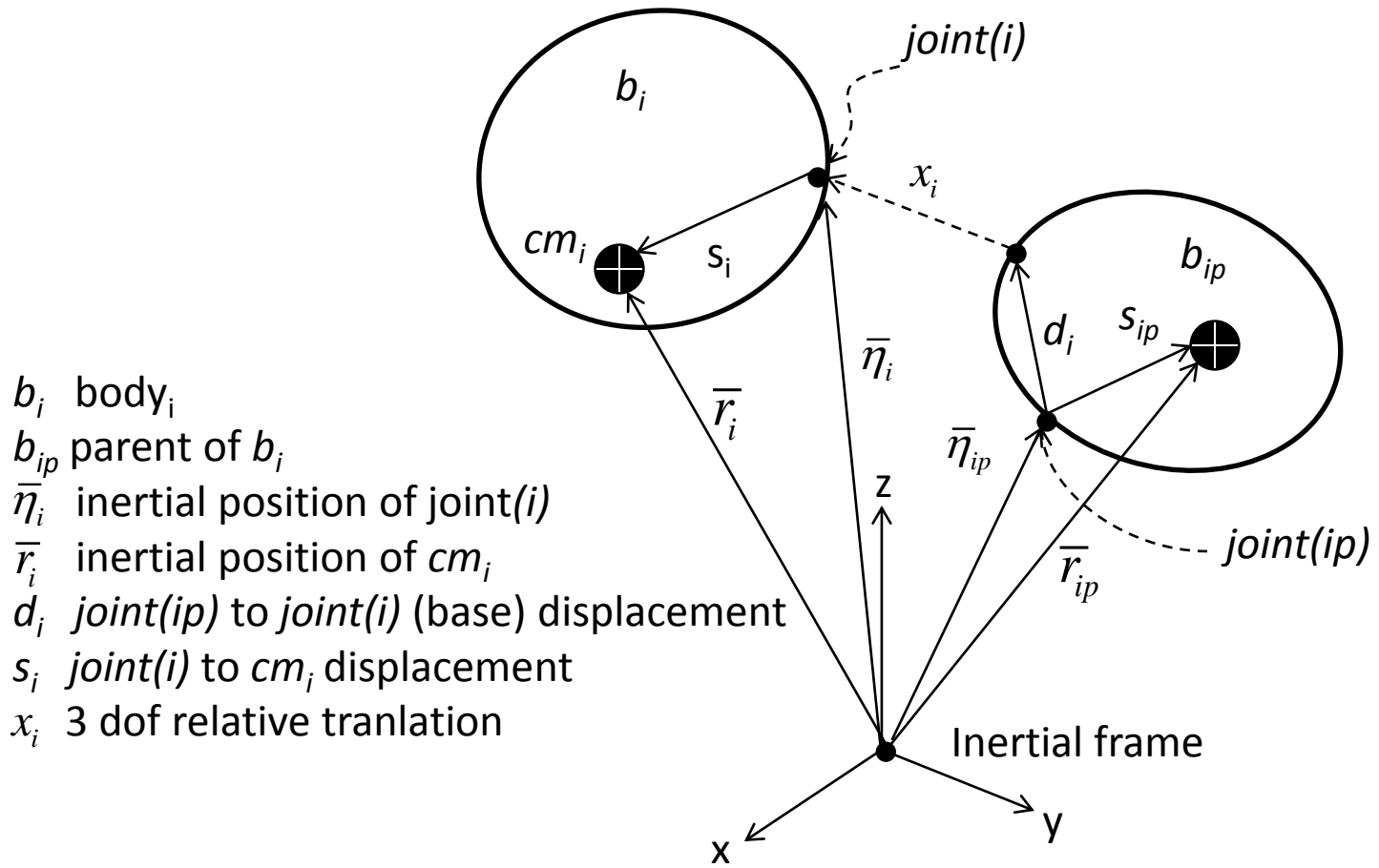


- The root body  $b_1$  is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. Generally, the choice of  $b_1$  is arbitrary. In the case of a humanoid robot,  $b_1$  could be the head, the torso or the hip.
- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.
- The chain of bodies between  $b_1$  and  $b_j$  shall be denoted as  $\{i \mid i \leq j\}$  or just  $i \leq j$ . The less-than-or-equal relation over body indices is a topological order and not a numerical order.
- The set of bodies branching from  $b_j$  shall be denoted as  $\{i \mid i \geq j\}$  or just  $i \geq j$ . The greater-than-or-equal relation over body indices is a topological order and not a numerical order.
- All vectors in the following discussion are given in the format  $x_j^i$ . The subscript  $j$  denotes the body that  $x$  belongs to and the superscript  $i$  denotes the coordinate frame that the vector is in.
- Vectors with no superscript are given in inertial coordinates unless defined otherwise

# Fig. 2 Prismatic Joint, JT=1

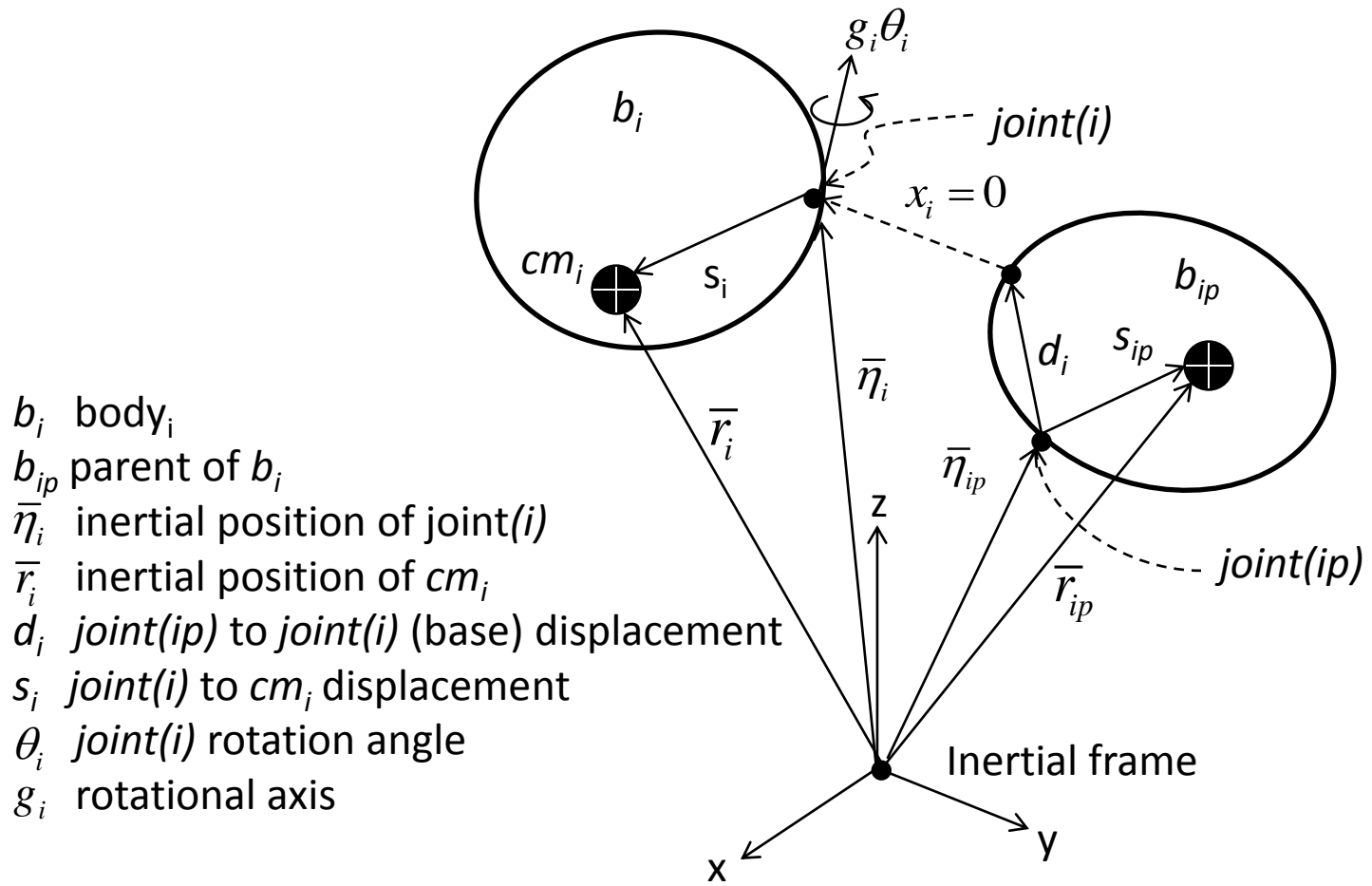


# Fig. 3 3 Dof Translational Joint, JT=2

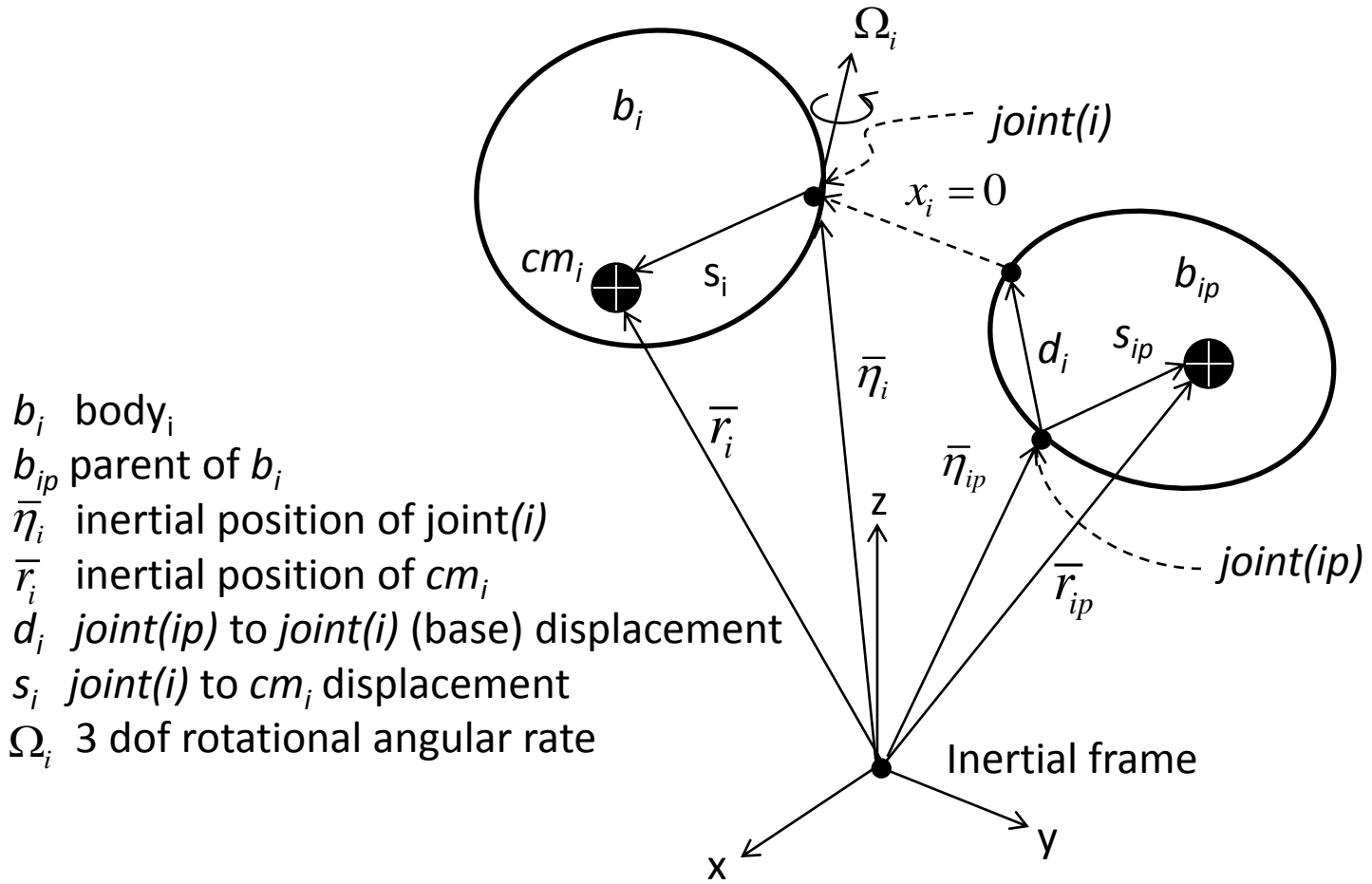




# Fig. 4 1 Dof Rotational Joint, JT=3



# Fig. 5 3 Dof Rotational Joint, JT=4



# Coordinate Frames

- A general reference frame is a 3 dimensional space with its origin defined at some point of interest. The translational motion of objects are defined by the  $[x, y, z]$  coordinates of the objects over time in that frame. The attitude motion of each body is defined by a direction cosine matrix that maps vectors fixed on that body to the reference frame over time.
- A local reference frame (*TRF*) shall mean a body fixed reference frame whose origin is located at the inboard joint of the body.
- An inertial frame is any frame in which Newtonian dynamics hold.
- Workspace frame is an inertial reference frame chosen to define the motion of a mechanism or an object such as a robot in that frame.

# Attitude Equations

- A local reference frame is a body fixed three dimensional coordinate frame with its origin at the inboard joint of the body. For  $b_1$ , the  $LRF_1$  origin is defined at an arbitrary reference point on it, called joint(1) here. See Fig. 1.
- Given  $q = \{\gamma_i\}_{i=1:N}$  the direction cosine matrix  $C_i$  from  $b_i$  to the inertial frame is defined recursively by

$$C_i = C_{ip} C_i^{ip}(\gamma_i) \text{ for } i \geq 2 \quad (2)$$

where  $C_i^{ip}(\gamma_i)$  = relative dcm from  $b_i$  to  $b_{ip}$  frame

$ip$  = index of parent body of  $b_i$

$C_1 = C_1^0$ , 0 superscript means the inertial frame

- If joint( $i$ ) is translational (JT( $i$ )=1 and 2)

$$C_i = C_{ip} \text{ and } C_i^{ip}(\gamma_i) = e_{3 \times 3} \quad (2a)$$

- If joint( $i$ ) is 1 dof rotational with  $\gamma_i = \theta_i$  (JT( $i$ )=3), then

$$C_i^{ip}(\theta_i) = \bar{C}_i^{ip} \left[ (1 - \cos(\theta_i)) g_i^i g_i^{iT} + \cos(\theta_i) e + \sin(\theta_i) \tilde{g}_i^i \right] \quad (2b)$$

where  $\bar{C}_i^{ip} = C_i^{ip}(\theta_i = 0)$

$g_i^i$  = joint( $i$ ) rotation axis in  $LRF_i$  coordinates

$\tilde{g}_i^i$  = skew symmetric matrix of vector  $g_i^i$

$e = 3 \times 3$  identity matrix

- If joint( $i$ ) is 3 dof rotational with the relative quaternion  $\gamma_i = \tilde{\varepsilon}_i = (a \ b \ c \ d)^T$ , (JT( $i$ )=4) then

$$C_i^{ip}(\tilde{\varepsilon}_i) = \begin{bmatrix} a^2 - b^2 - c^2 + d^2 & 2(ab - cd) & 2(ac + bd) \\ 2(ab + cd) & -a^2 + b^2 - c^2 + d^2 & 2(bc - ad) \\ 2(ac - bd) & 2(bc + ad) & -a^2 - b^2 + c^2 + d^2 \end{bmatrix} \quad (2c)$$

- Given  $q = \{\gamma_i\}_{i=1:N}$ , all  $\{C_i(\gamma_i)\}_{i=1:n}$  can be computed by Eqs. (2,2a,2b,2c).

# Position Equations

- Given  $\{C_i(\gamma_i)\}_{i=1:n}$ , all body fixed vectors in the system can be expressed in inertial coordinates
- All joint and cm positions  $\{\bar{\eta}_i, \bar{r}_i\}_{i=1:n}$  in the inertial reference frame per Figs. 1 and 2 can be computed as

$$\bar{\eta}_i = \begin{cases} \bar{\eta}_{ip} + d_i + \hat{g}_i \delta_i & , \text{ if } \gamma_i = \delta_i \\ \bar{\eta}_{ip} + d_i + C_{ip} \mathbf{x}_i^i & , \text{ if } \gamma_i = \mathbf{x}_i^i \\ \bar{\eta}_{ip} + d_i & , \text{ if } \gamma_i = \theta_i \text{ or } \bar{q}_i \end{cases} \quad (3)$$

for  $i = 2 : N$  with  $\bar{\eta}_{1p} = 0$

$$\bar{r}_i = \bar{\eta}_i + s_i \quad , \text{ for } i = 1 : N \quad (4)$$

where  $d_i = \bar{\eta}_i$  position from  $\bar{\eta}_{ip}$ ;  $\hat{g}_i =$  slider axis;  $\delta_i =$  linear displacement

$s_i = cm_i$  position from  $\bar{\eta}_i$ ;  $\mathbf{x}_i^i =$  displacement vector in  $b_i$  coordinates

$\bar{\eta}_i =$  inertial position of joint( $i$ )

$\bar{r}_i =$  inertial position of  $cm_i$

# Velocity Equations

- The system rates state  $\dot{q}^* = \{\dot{\gamma}_i^*\}_{i=1:N}$  is either obtained by solving the equations of motion or prescribed, with

$$\dot{\gamma}_i^* = \begin{cases} \dot{\delta}_i & \text{if } \gamma_i = \delta_i (=> \Omega_i^i = 0) ; \text{JT}=1 \\ \dot{x}_i^i & \text{if } \gamma_i = x_i^i (=> \Omega_i^i = 0) ; \text{JT}=2 \\ \dot{\theta}_i & \text{if } \gamma_i = \theta_i (=> \dot{x}_i^i = 0) ; \text{JT}=3 \\ \Omega_i^i & \text{if } \gamma_i = \bar{q}_i (=> \dot{x}_i^i = 0) ; \text{JT}=4 \end{cases}$$

$\Omega_i^i =$  angular rate of  $LRF_i$  w.r.t  $LRF_{ip}$  in  $LRF_i$  coordinates



- $\tilde{\varepsilon}_i$  relates to  $\Omega_i^i$  for JT=4 as

$$\tilde{\varepsilon}_i = \frac{1}{2} E_i \Omega_i^i \quad (5)$$

where  $\tilde{\varepsilon}_i = [a, b, c, d]^T$  and  $E_i = \begin{bmatrix} d & -c & b \\ c & d & -a \\ -b & a & d \\ -a & -b & -c \end{bmatrix}$

- Given  $\dot{q}^* = \{\dot{\gamma}_i^*\}_{i=1:N}$  all body angular velocities can be computed by

$$\omega_i = \begin{cases} \omega_{ip} & \text{JT}(i)=1 \text{ or } 2 \\ \omega_{ip} + g_i \dot{\theta}_i & \text{JT}(i)=3 \\ \omega_{ip} + \Omega_i & \text{JT}(i)=4 \end{cases} \quad (6)$$

for  $i = 1:N$ , with  $\omega_{1p} = 0$

where  $g_i =$  inboard 1 dof joint rotational axis of  $b_j$

$$\Omega_i = C_i \Omega_i^i ; \text{ relative angular rate in inertial coordinates} \quad (7)$$

- Given  $\{\dot{\hat{\eta}}_1, \omega_i\}_{i=1:n}$  the inertial velocities of all joints can be computed per Figs. 1 and 2 as

$$\dot{\hat{\eta}}_i = \begin{cases} \dot{\hat{\eta}}_{ip} + \omega_{ip} \times d_i + \dot{\delta}_i \hat{g}_i & \text{JT}(i)=1 \\ \dot{\hat{\eta}}_{ip} + \omega_{ip} \times d_i + C_{ip} \dot{x}_i^i & \text{JT}(i)=2 \\ \dot{\hat{\eta}}_{ip} + \omega_{ip} \times d_i & \text{JT}(i)=3 \text{ or } 4 \end{cases} \quad (8)$$

for  $i = 2 : N$  with  $\dot{\hat{\eta}}_{1p} = 0$

where

$$d_i = \begin{cases} C_{ip} (d_i^i + \hat{g}_i \delta_i) & \text{JT}(i)=1 \\ C_{ip} (d_i^i + x_i^i) & \text{JT}(i)=2 \\ C_{ip} d_i^i & \text{JT}(i)=3 \text{ or } 4 \end{cases}$$

- Given  $\{\dot{\hat{\eta}}_i, \omega_i\}_{i=1:n}$  the inertial velocities of all body  $cm$ 's can be computed follows

$$\dot{\hat{r}}_i = \dot{\hat{\eta}}_i + \omega_i \times s_i, \text{ for } i = 1 : N \quad (9)$$

# Acceleration Equations

- Given  $\{\dot{\omega}_1, \dot{\gamma}_i^*\}_{i=1:N}$  all angular accelerations can be computed recursively as

$$\dot{\omega}_i = \begin{cases} \dot{\omega}_{ip} & \text{JT}(i)=1 \text{ or } 2 \\ \dot{\omega}_{ip} + g_i \ddot{\theta}_i + \omega_{ip} \times g_i \dot{\theta}_i & \text{JT}(i)=3 \\ \dot{\omega}_{ip} + C_i \dot{\Omega}_i^i + \omega_{ip} \times C_i \Omega_i^i & \text{JT}(i)=4 \end{cases} \quad (10)$$

for  $i = 2:N$

- Given  $\{\ddot{\eta}_1, \dot{\omega}_i\}_{i=1:N}$  and Eqs. (7, 8, 9) all joint accelerations can be computed recursively as

$$\ddot{\eta}_i = \begin{cases} \ddot{\eta}_{ip} + \dot{\omega}_{ip} \times (d_i + \delta_i \hat{g}_i) + \ddot{\delta}_i \hat{g}_i + 2(\omega_{ip} \times \dot{\delta}_i \hat{g}_i) + \omega_{ip} \times (\omega_{ip} \times (d_i + \delta_i \hat{g}_i)) & \text{(JT}(i)=1) \\ \ddot{\eta}_{ip} + \dot{\omega}_{ip} \times (d_i + \mathbf{x}_i) + C_i \ddot{\mathbf{x}}_i^i + 2(\omega_{ip} \times \dot{\mathbf{x}}_i) + \omega_{ip} \times (\omega_{ip} \times (d_i + \mathbf{x}_i)) & \text{(JT}(i)=2) \\ \ddot{\eta}_{ip} + \dot{\omega}_{ip} \times d_i + \omega_{ip} \times (\omega_{ip} \times d_i) & \text{(JT}(i)=3 \text{ or } 4) \end{cases} \quad (11)$$

for  $2 = 1:n$

- Given  $\{\dot{\eta}_i, \omega_i\}_{i=1:n}$ , the cm inertial accelerations can be computed as

$$\ddot{r}_i = \ddot{\eta}_i + \dot{\omega}_i \times s_i + \omega_i \times (\omega_i \times s_i) \quad \text{for } i = 1:n \quad (12)$$

# Other Points of Interest

- Motion of position markers on bodies may be needed for performance or constraint evaluations.
- Calculating the  $k$ -th position marker on  $b_j$  for some  $j$  :

$$p_j(k) = C_j p_j^j(k) \quad (13)$$

$$\bar{p}_j(k) = \bar{\eta}_j + p_j(k) \quad (14)$$

$$\dot{\bar{p}}_j(k) = \dot{\bar{\eta}}_j + \omega_j \times p_j(k) \quad (15)$$

$$\ddot{\bar{p}}_j(k) = \ddot{\bar{\eta}}_j + \dot{\omega}_j \times p_j(k) + \omega_j \times (\omega_j \times p_j(k)) \quad (16)$$

where  $p_j^j(k)$  = position of marker( $k$ ) from joint( $j$ ) in  $b_j$  coordinates

$p_j(k)$  = inertial coordinates of  $p_j^j(k)$

$\bar{p}_j(k)$  = inertial position of position marker( $k$ )

$\dot{\bar{p}}_j(k)$  = inertial velocity of position marker( $k$ )

$\ddot{\bar{p}}_j(k)$  = inertial acceleration of position marker( $k$ )

- Motion of directional markers on member bodies may also be needed for performance or in constraint evaluations.
- Calculating the  $k$ -th directional marker on  $b_j$  for some  $j$ :

$$u_j(k) = C_j u_j^j(k) \quad (17)$$

$$\dot{u}_j(k) = \omega_j \times u_j(k) \quad (18)$$

$$\ddot{u}_j(k) = \dot{\omega}_j \times u_j(k) + \omega_j \times (\omega_j \times u_j(k)) \quad (19)$$

where  $u_j^j(k)$  = unit vector of direction marker( $k$ ) on  $b_j$  in  $b_j$  coordinate

$u_j(k)$  = inertial coordinates of  $u_j^j(k)$

$\dot{u}_j(k)$  = inertial velocity (turning rate) of  $u_j(k)$

$\ddot{u}_j(k)$  = inertial acceleration of  $u_j(k)$

# Examples

- A ball joint is one with the coordinate  $\gamma_j = \bar{q}_j$ .
- A slider joint is one with the coordinate  $\gamma_j = \delta_j \in \mathbb{R}$
- A universal joint consists of two serially linked joints of  $b_{jp}$  and  $b_j$  such that

$$(\gamma_{jp}, \gamma_j) = (\theta_{jp}, \theta_j) \text{ with } g_j^T g_{jp} = 0 \text{ and } d_j = 0_{3 \times 1}$$

- A 6 dof joint consists of two serial joints of  $b_{jp}$  and  $b_j$  such that

$$(\gamma_{jp}, \gamma_j) = (x_{jp}^{jp}, \bar{q}_j), \text{ with } d_j = 0$$



# Summary

- The kinematics equations in attitude, position, velocity and acceleration have been presented for a rigid joint-connected multibody system that has different types of joints. These joints can have either 1 or 3 dofs and are either rotational or translational. The kinematic equations are defined recursively and conditionally based on the type of joint.
- Kinematics equations for the position and directional markers have been presented for the purpose of performance and constraint evaluations.