EOM with Hamilton's Equations

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Content

Introduction

- Hamilton's equations are a modified form of the *Euler Lagrange* equations, in that a Hamiltonian function is used to arrive at a system of $2N$ first order differential equations or equations of motion (EOM) in (q, p); q is the generalized coordinates and p is the generalized momenta of the system.
- Solution to Hamilton's equations based EOM requires a rate solution by $\dot{q} = \mathcal{M}^{-1} p$ in order to propagate the pair (q, p) forward in time, where $\mathcal{M}$ is the system mass matrix.
- In contrast, the EOM by the EL equations is a system of N second order differential equations in the generalized coordinates q. The generalized acceleration, $\ddot{q}$, is solved from these EOM and double integrated to propagate the state of the MBS, in this case $(q, \dot{q})$, forward in time.

- Tree-structured rigid multibody systems (MBS) that have only single axis rotational motion between joint connected bodies are considered to expedite the discussion. The root body is connected to the ground in all cases considered. These systems encompass robots, mechanical arms, multi-fingered hands, cranes, gimbaled antennas, transmission and suspension systems.

- Generalized coordinates and system rates for these MBS are

$$q = \{\theta_i\}_{i=1:N}, \ \dot{q} = \{\dot{\theta}_i\}_{i=1:N}$$

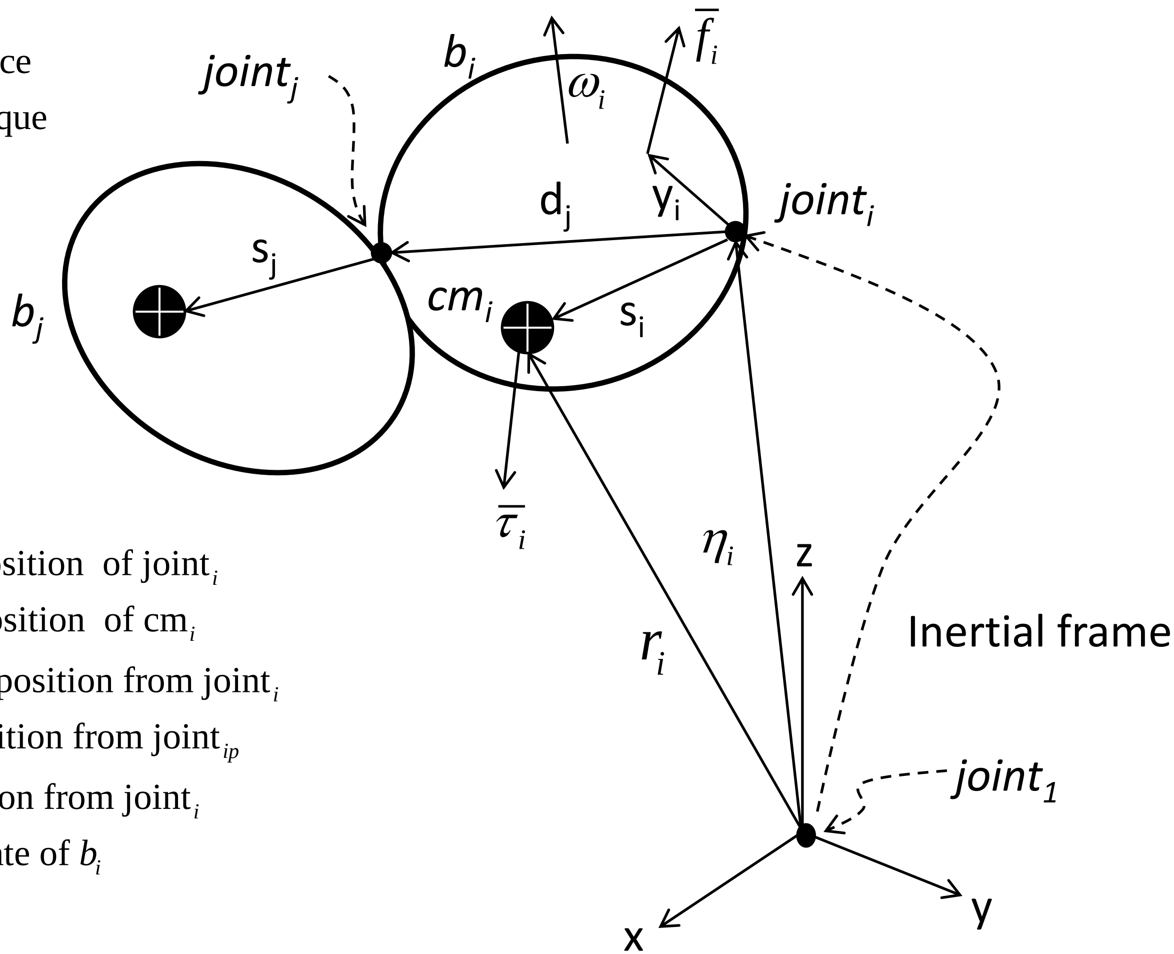
where $\theta_i =$ inboard joint angle of b_i

- Ensuing discussions cover 1) dynamics equations of an MBS, 2) generalized momenta p, and 3) solving $\dot{q}$ given p.

Fig. 1 Notations

$\overline{f}_i$ external force

$\overline{\tau}_i$ external torque



b_i body$_i$

η_i inertial position of joint$_i$

r_i inertial position of cm$_i$

y_i $\overline{f}_i$ impact position from joint$_i$

d_i joint$_i$ position from joint$_{ip}$

s_i cm$_i$ position from joint$_i$

ω_i angular rate of b_i

- The root body b_1 is the root body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. This root body here is connected to the ground

- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.

- The chain of bodies between b_1 and b_j shall be denoted as $\{i \mid i \leq j\}$ or just $i \leq j$.
  The less-than-or-equal relation over body indices is a topological order and not a numerical order.

- The set of bodies branching from b_j shall be denoted as $\{i \mid i \geq j\}$ or just $i \geq j$.
  The greater-than-or-equal relation over body indices is a topological order and not a numerical order.

- All vectors in the following discussion are given in the format x_j^i . The subsript j denotes the body that x belongs to and the superscript i denotes the coordinate frame that the vector is in.

- Vectors with no superscript are given in inertial coordinates unless defined otherwise

- A stacked vector y is a column vector whose elements are also vectors. The latter can be of different sizes. We define this stacked vector as

$$y=[y_1, y_2, y_3, \cdots, y_N], y_i \text{ is a vector}$$

where $\text{size}(y)=len(y)\times 1, len(y)=\sum_{i=1}^{N} len(y_i)$

For example:

Let $y_1 = v_x$, $y_2 = \begin{bmatrix} s_x \\ s_y \end{bmatrix}$, $y_3 = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$

Then $y=[y_1, y_2, y_3]=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} v_x \\ s_x \\ s_y \\ w_x \\ w_y \\ w_z \end{bmatrix}$

- Skew matrix notation: $\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$, if $a = \left[a_x, a_y, a_z \right]$

- $e = k \times k$ identity matrix, size k depends on context

- The relations $\{ >, \ <, \ \geq, \ \leq \}$ shall mean topological order in the following expressions when used to scope body indices, unless stated otherwise.

- $\overline{i} \ = \ \{\alpha \,|\, parent(\alpha) = i\}$, indices of children of b_i

- $ip = \ parent(i)$, index of parent of b_i

- Each body in a tree-configured mechanism has one parent body except for b_1. Thus, each row of $\Phi_{\mathcal{A}}$ has only one non-zero submatrix entry and the first row is all zeros.

- Given that $\Phi_{\mathcal{A}}$ is square and strictly triangular, it is nilpotent of degree *m*, i.e.

$$\Phi_{\mathcal{A}}^m = 0$$

where, $m =$ length of the longest link in the system from b_1, and $m \le N$.

- Let $\mathcal{A} = diag\{\mathcal{A}_i\}_{i=1:N}, \mathcal{A}_i \in R^{k\times k}, \mathcal{A}_1$=0, and $x = col[x_i],\ z = col[z_i]$, with $x_i, z_i \in R^k, k > 1.$

  Prop1. $z = \Phi_{\mathcal{A}} x$ is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = \mathcal{A}_i x_{ip} \text{ for } i = 1:N \qquad \text{(P1)}$$

  Prop2. $z = \Phi_{\mathcal{A}}^T x$ is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j\in\bar{i}} \mathcal{A}_j^T x_j \text{ for } i = 1:N \qquad \text{(P2)}$$

Influence Matrices Φ_A and Φ_A^T

- Influence matrices , Φ_A and Φ_A^T , are defined by the parent child relations of the considered mechanism and a block diagonal matrix A:

$$[\Phi_A]_{i,j} = \begin{cases} A_i \text{ if } & b_j = \text{parent of } b_i \\ 0_{6\times6} \text{ otherwise} \end{cases} \qquad \text{Def. 1}$$

$$[\Phi_A^T]_{i,j} = \begin{cases} A_j^T \text{ if } & b_i = \text{parent of } b_j \\ 0_{6\times6} \text{ otherwise} \end{cases} \qquad \text{Def. 2}$$

where $A = \text{diag}\{A_j\}_{j=1:N}$, a block diagonal matrix, with $A_i \in \mathrm{R}^{k\times k}, A_1 = 0$

$k = \text{integer} > 1$

- Forward influence matrix, Φ_A is strictly lower triangular.
- Backward influence matrix, Φ_A^T is strictly upper triangular.

- Each body in a tree-configured mechanism has one parent body except for b_1. Thus, each row of $\Phi_{\mathcal{A}}$ has only one non-zero submatrix entry and the first row is all zeros.

- Given that $\Phi_{\mathcal{A}}$ is square and strictly triangular, it is nilpotent of degree m, i.e.

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Prop1. $z = \Phi_{\mathcal{A}} x$ is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = \mathcal{A}_i x_{ip} \text{ for } i = 1:N \tag{P1}$$

Prop2. $z = \Phi_{\mathcal{A}}^T x$ is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j \in \bar{i}} \mathcal{A}_j^T x_j \text{ for } i = 1:N \tag{P2}$$

- If $\Phi_{\mathcal{A}}$ is square and strictly lower triangular, then $\Psi_{\mathcal{A}} = (e - \Phi_{\mathcal{A}})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}} = e + \Phi_{\mathcal{A}} + \Phi_{\mathcal{A}}^{2} + \cdots + \Phi_{\mathcal{A}}^{m-1} \tag{P3}$$

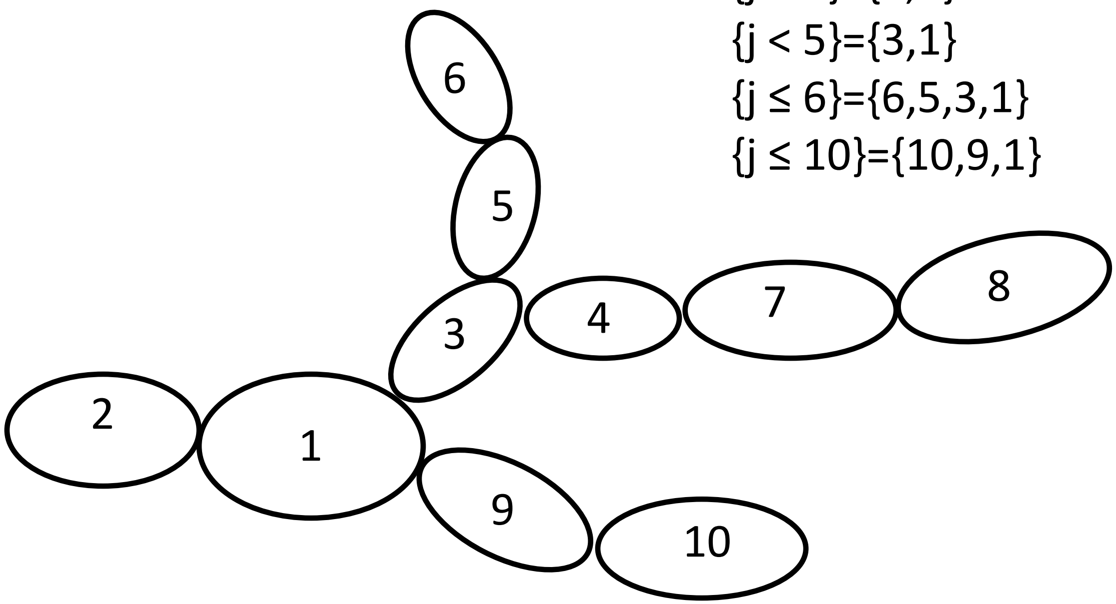
where $m =$ nilpotency of Φ_{A}

- It follows that $\Phi_{\mathcal{A}}^{T}$ is square and strictly upper triangular, and $\Psi_{\mathcal{A}}^{T} = (e - \Phi_{\mathcal{A}}^{T})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}}^{T} = e + \Phi_{\mathcal{A}}^{T} + \Phi_{\mathcal{A}}^{T,2} + \cdots + \Phi_{\mathcal{A}}^{T,m-1} \tag{P4}$$

- The power series expansion of $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}^{T}$ serve to show the existence of $(e + \Phi_{\mathcal{A}})^{-1}$ and $(e + \Phi_{\mathcal{A}}^{T})^{-1}$. More efficient way of computing $\Psi_{\mathcal{A}} x$ and $\Psi_{\mathcal{A}}^{T} x$ for any $x \in \mathrm{R}^{6N}$ is shown shortly.

Fig. 2 Example Body Sets



10 body example

$\{j \le 2\}=\{2,1\}$
$\{j < 5\}=\{3,1\}$
$\{j \le 6\}=\{6,5,3,1\}$
$\{j \le 10\}=\{10,9,1\}$

$branch_1=\{j \ge 1\}=\{1:10\}$
$branch_3=\{j \ge 2\}=\{2\}$
$branch_3=\{j \ge 3\}=\{3:8\}$
$branch_4=\{j \ge 4\}=\{4,7,8\}$

$$\overline{1}=\{2,3,9\}$$
$$\overline{2}=\overline{6}=\overline{8}=1\overline{0}=\{empty\}$$
$$\overline{3}=\{4,5\}$$
$$\overline{4}=\{7\}$$

Φ_A For Fig. 2 Example

$$\Phi_A = \begin{bmatrix} 0 & & & & & & & & & \\ A_2 & 0 & & & & & & & & \\ A_3 & 0 & 0 & & & & -0- & & & \\ 0 & 0 & A_4 & 0 & & & & & & \\ 0 & 0 & A_5 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & A_6 & 0 & & & & \\ 0 & 0 & 0 & A_7 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & A_8 & 0 & & \\ A_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10} & 0 \end{bmatrix}$$

Ψ_A For Fig. 2 Example

If A_i is a square submatrix $\forall\, i$ for this example, then

$$\Psi_A = \left(e - \Phi_A\right)^{-1} = \begin{bmatrix} e & & & & & & & & & \\ A_2 & e & & & & & & & & \\ A_3 & 0 & e & & & & -0- & & & \\ A_4A_3 & 0 & A_4 & e & & & & & & \\ A_5A_3 & 0 & A_5 & 0 & e & & & & & \\ A_6A_5A_3 & 0 & A_6A_5 & 0 & A_6 & e & & & & \\ A_7A_4A_3 & 0 & A_7A_4 & A_7 & 0 & 0 & e & & & \\ A_8A_7A_4A_3 & 0 & A_8A_7A_4 & A_8A_7 & 0 & 0 & A_8 & e & & \\ A_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & \\ A_{10}A_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10} & e \end{bmatrix}$$

nilpotency(Φ_A)=5

O(N) Recursive Algorithms

- Examples presented earlier showed the lower-triangular nature of $\Psi_{\mathcal{A}}$ given the parent-child relation of bodies in a tree-configured system.

- Computing expressions involving $\left(\Psi_{\mathcal{A}} x\right)$ or $\left(\Psi^{T}_{\mathcal{A}} x\right)$ by direct matrix-vector multiplication (i.e. using (P3) and (P4) for $\Psi_{\mathcal{A}}$ and $\Psi^{T}_{\mathcal{A}}$) is inefficient.

- The next two algorithms are order(N) computations of those expressions.

Algorithm for $z = \Psi_{\mathcal{A}} x$

- A stacked vector given by $z = \Psi_{\mathcal{A}} x$ is the result of solving $z = \Phi_{\mathcal{A}} z + x$ for z, where $\Phi_{\mathcal{A}}$ is square and strictly lower triangular per Def 1. Thus, given x, the elements of z are obtained by the algorithm below :

  1. set $z_1 = x_1$ since the first row of $\Phi_{\mathcal{A}}$ is zero
  2. for $i = 2 : N$

     $z_i = \mathcal{A}_i z_{ip} + x_i \quad ;$

     end

- The above computation of z given x is order(N). Thus $\Psi_{\mathcal{A}}$ is an order(N) operator.

Algorithm for $z = \Psi_{\mathcal{A}}^{T} x$

- A stacked vector given by $z = \Psi_{\mathcal{A}}^{T} x$ is the result of solving $z = \Phi_{\mathcal{A}}^{T} z + x$ for z, where $\Phi_{\mathcal{A}}^{T}$ is square and strictly upper triangular. Thus, given x, the elements of z are obtained by the algorithm below :

  1. set $z = x$ since the first column of $\Phi_{\mathcal{A}}^{T}$ is zero
  2. for $i = N : 2$

     $z_{ip} := z_{ip} + \mathcal{A}_i^T z_i \quad ;$

     end

- The above computation of z given x is order(N). Thus $\Psi_{\mathcal{A}}^{T}$ is an order(N) operator.

Velocity Equations

- Relation between $\overline{v}_i$ and v_i is (See Fig. 1)

$$\overline{v}_i = S_i v_i \qquad \text{(V.1)}$$

where $\overline{v}_i = [\omega_i, \dot{r}_i]$, $v_i = [\omega_i, \dot{\eta}_i]$ and $\omega_i =$ total angular rate of b_i

$\dot{r}_i =$ velocity of cm_i, $\dot{\eta}_i =$ velocity of joint$_i$

$$S_i = \begin{bmatrix} e & 0 \\ -\tilde{s}_i & e \end{bmatrix}, \; s_i = \text{ displacement from joint}_i \text{ to } cm_i$$

- v_i relates to v_{ip} and the relative rate $\dot{q}_i$ by

$$v_i = D_i v_{ip} + G_i \dot{q}_i \qquad \text{(V.2)}$$

where $D_i = \begin{bmatrix} e & 0 \\ -\tilde{d}_i & e \end{bmatrix}_{i=2:N}, D_1 = 0, v_{1p} = 0,$

$$G_i = [g_i, 0]_{i=1:N}, \; \dot{q} = \left[\dot{\theta}_i\right]_{i=1:N}$$

- Given Def. 1 and Prop.1, Eq. (V.2) can be put in column vector form as

$$v = \Phi_D v + G\dot{q} \tag{V.3}$$

where, $v = col[v_i]_{i=1:N}, D = diag[D_i]_{i=1:N}, G = diag[G_i]_{i=1:N}, \dot{q} = \left[\dot{\theta}_i\right]_{i=1:N}$

- The velocity vector v in (V.3) can be solved to be

$$v = \Psi_D G\dot{q} \tag{V.4}$$

where $\Psi_D = (e - \Phi_D)^{-1}$

- Given (V.1) and (V.4),

$$\bar{v} = S\Psi_D G\dot{q} \tag{V.5}$$

where, $\bar{v} = col[\bar{v}_i]_{i=1:N}, S = diag[S_i]_{i=1:N}$

Hamilton's Equations

- Hamilton's equations for a N-body system are $2N$ equations given by (Ref. [3]).

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial q} + Q \tag{2}$$

where $H = \dot{q}^T p - L$; Hamiltonian (3)

$L = T(q, \dot{q}) - V(q)$; Lagrangian (4)

$T =$ kinetic energy

$V =$ potential energy

$$p = \frac{\partial L}{\partial \dot{q}} \quad \text{; generalized momenta} \tag{5}$$

$Q =$ generalized forces

- The total kinetic energy of a rigid multibody system is

$$T = \tfrac{1}{2} \sum_{i=1:N} \left(\omega_i^T I_i \omega_i + m_i \dot{r}_i^T \dot{r}_i \right) \tag{6}$$

- It can also be expressed as

$$T = \tfrac{1}{2} \dot{q}^T \mathcal{M} \dot{q} \qquad \text{(See generalized mass matrix presentation)} \tag{7}$$

where $\mathcal{M} = G^T \Psi_D^T M \Psi_D G,$ system mass matrix (see (P.4),(P.4a))

$\dot{q} = [\dot{\theta}_i]_{i=1:N}$

$\omega_1 =$ total angular rate of b_1

$\dot{\eta}_1 = 0 =$ inertial velocity of reference point on b_1

$\dot{\theta}_i =$ relative angular rate of joint_i

- Equation (2) is essentially the Euler – Lagrange equation when p in (2) is replaced by (5) and

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \qquad \text{(for } i = 1:N, \text{per (3))} \tag{8}$$

Generalized Momenta

- Generalized momenta per Eqs. (4), (5) and (7) are

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = \mathcal{M}\dot{q} \tag{9}$$

where, $p = [p_{\theta_1}, \cdots, p_{\theta_N}]$

$\dot{q} = [\dot{\theta}_1, \cdots, \dot{\theta}_N]$

- It follows that (1) is the solution of (9), such that

$$\dot{q} = \frac{\partial H}{\partial p} = \mathcal{M}^{-1} p \tag{10}$$

- Given (10), the kinetic energy (7) can also be written as

$$T = \tfrac{1}{2} p^T \mathcal{M}^{-1} p \tag{11}$$

- Proposition 1a. The generalized momentum with respect to ω_j is

$$H_i = \frac{\partial L}{\partial \omega_i} = \sum_{k \ge i} \left(I_k \omega_k + m_k (r_k - \eta_i) \times \dot{r}_k \right) \qquad \text{(P.1a)}$$

Proof. Take the partial of Eq. (6) w.r.t. ω_i to obtain (P.1a). QED

- Proposition 1b. The generalized momentum conjugate to η_i is

$$P_i = P_{\eta_i} = \frac{\partial L}{\partial \dot{\eta}_i} = \sum_{k \ge i} m_k \dot{r}_k \qquad \text{(P.1b)}$$

Proof. Take the partial of Eq. (6) w.r.t. $\dot{\eta}_i$ to obtain (P.1b). QED

- The center of mass of $\{b_k \mid k \ge i\}$ measured from η_i is defined by

$$c_i = \frac{1}{\bar{m}_i} \sum_{k \ge i} m_k (r_k - \eta_i) \qquad \text{(P.1c)}$$

where, $\bar{m}_i = \sum_{k \ge i} m_k$

- $$\dot{c}_i = \frac{1}{\bar{m}_i}\sum_{k \geq i} m_k(\dot{r}_k - \dot{\eta}_i) = \frac{P_i}{\bar{m}_i} - \dot{\eta}_i \qquad \text{(P.1d)}$$

- Newton's 2nd law of motion for $\{b_k \mid k \geq i\}$, all bodies outboard connected to b_i,

$$\dot{P}_i = f_i + \sum_{k \geq i} \bar{f}_k \qquad \text{(P.1e)}$$

where $f_i =$ interbody force from b_{ip} to b_i at its inboard joint

$\bar{f}_k =$ external force on b_k

- Euler equation of the total angular momentum rate of $\{b_k \mid k \geq i\}$ about the cm of that set is

$$\dot{H}_i^c = \tau_i - c_i \times f_i + \sum_{k \geq i}\left(\bar{\tau}_k + (\bar{y}_k - \eta_i - c_i) \times \bar{f}_k\right) \qquad \text{(P.1f)}$$

where, $H_i^c = \sum_{k \geq i}\left(I_k \omega_k + m_k(r_k - \eta_i - c_i) \times \dot{r}_k\right)$

$\bar{y}_k = \eta_k + y_k$; $\bar{\tau}_k =$ external torque on b_k

$\tau_i =$ interbody torque from b_{ip} to b_i at its inboard joint

$c_i =$ cm of $\{b_k \mid k \geq i\}$ from joint$_i$

- Proposition 1g.

$$\dot{H}_i = -\dot{\eta}_i \times P_i + \tau_i + \sum_{k \geq i} \left(\overline{\tau}_k + (\overline{y}_k - \eta_i) \times \overline{f}_k \right) \qquad \text{(P.1g)}$$

Proof. Show that $H_i = H_i^c + c_i \times P_i$. Take the time derivative of H_i and apply (P.1c) to (P.1f) to the resulting equation and obtain

$$\begin{aligned} \dot{H}_i &= \tau_i - c_i \times f_i + \sum_{k \geq i} (\overline{\tau}_k + (\overline{y}_k - \eta_i - c_i) \times \overline{f}_k) + \dot{c}_i \times P_i + c_i \times (f_i + \sum_{k \geq i} \overline{f}_k) \\ &= \dot{c}_i \times P_i + \tau_i + \sum_{k \geq i} (\overline{\tau}_k + (\overline{y}_k - \eta_i) \times \overline{f}_k) \\ &= -\dot{\eta}_i \times P_i + \tau_i + \sum_{k \geq i} (\overline{\tau}_k + (\overline{y}_k - \eta_i) \times \overline{f}_k) \end{aligned}$$

Last step of the above $\dot{H}_i$ reduction holds because Eq. (P.1b) reduces to $P_i = \overline{m}_i (\dot{c}_i + \dot{\eta}_i)$. QED

- Proposition 2. System momentum vector is

$$\mathscr{H} = \Psi_D^T M v \qquad \text{(P.2)}$$

where $\mathscr{H} = [\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_N]; \mathscr{H}_i = [H_i, P_i]$

$D = diag[D_i]_{i=1:N}, D_1 = 0$

$\Psi_D^T =$ backward recursive operator, see Newton-Euler Eq. presentation

$M = diag[M_i]_{i=1:N}, M_i = \begin{bmatrix} I_i - m_i \tilde{s}_i \tilde{s}_i & m_i \tilde{s}_i \\ -m_i \tilde{s}_i & m_i e \end{bmatrix}$

$s_i =$ joint$_i$ to cm_i displacement vector

$v = [v_1, v_2, \cdots, v_N]$, with $v_i = [\omega_i, \dot{\eta}_i]$

Proof. Find $\mathscr{H}_i$ in a recursive form using Eqs. (P.1a) and (P.1b).

Apply Def.2 and Prop.2 to obtain the vector $\mathscr{H}$ in a recursive form.

Solve for $\mathscr{H}$ to obtain (P.2).

- Proposition 3a. Momentum state is

$$p = \frac{\partial L}{\partial \dot{q}} = G^T \mathcal{H} \qquad \text{(P.3a)}$$

where $\mathcal{H} = [\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N]; \mathcal{H}_j = [H_j, P_j]$

$$G = diag[G_i]_{i=1:N}, G_i = [g_i, 0_{3\times1}] = \begin{bmatrix} g_i \\ 0_{3\times1} \end{bmatrix}_{i=1:N}$$

$g_i =$ free motion axis of LRF_i

Proof. $p = \left[\frac{\partial L}{\partial \dot{q}_1}, \frac{\partial L}{\partial \dot{q}_2}, \cdots, \frac{\partial L}{\partial \dot{q}_N} \right]$

$$\frac{\partial L}{\partial \dot{q}_i} = p_{\theta_i} = \left[\frac{\partial \omega_i}{\partial \dot{\theta}_i} \right]^T \frac{\partial L}{\partial \omega_i} = g_i^T H_i = [g_i, 0]^T [H_i, P_i] = G_i^T \mathcal{H}_i \qquad \text{(P.3a1)}$$

for $i = 1:N$

(P.3a) is the stacked vector notation of (P.3a1) with $G^T = diag\left[G_i^T \right]_{i=1:N}$.

QED

- Proposition 3b. Time derivative of p_{θ_i} is

$$\dot{p}_{\theta_i} = \hat{\tau}_i + g_i^T \left[-\omega_i \times H_i - \dot{\eta}_i \times P_i + \sum_{k \geq i} \left[\overline{\tau}_k + (\overline{y}_k - \eta_i) \times \overline{f}_k \right] \right] \qquad \text{(P.3b)}$$

where, $\hat{\tau}_i = g_i^T \tau_i$; actuation torque across g_i.

Proof. Take the time derivative of (P.3a1) for each j to obtain

$$\begin{aligned} \dot{p}_{\theta_i} &= G_i^T [\dot{H}_i, \dot{P}_i] + \dot{G}_i^T [H_i, P_i] = g_i^T \dot{H}_i + (\omega_i \times g_i)^T H_i \\ &= g_i^T (\dot{H}_i - \omega_i \times H_i) \end{aligned}$$

Replace $\dot{H}_i$ in the above equation by (P.1g) to obtain (P.3b). QED

- Proposition 4. Momentum state of the system is

$$p = G^T \Psi_D^T M \Psi_D G\dot{q} \qquad \text{(P.4)}$$

where $p = [p_{\theta_1}, \cdots, p_{\theta_N}]$

$$G = diag[G_i]_{i=1:N}, G_1 = e_{6\times6}, G_i = [g_i, 0_{3\times1}]$$

$\Psi_D, \Psi_D^T =$ forward and backward recursive operators, see Newton-Euler Eq. presentation

$$M = diag[M_i]_{i=1:N}, M_i = \begin{bmatrix} I_i - m_i\tilde{s}_i\tilde{s}_i & m_i\tilde{s}_i \\ -m_i\tilde{s}_i & m_i e \end{bmatrix}$$

$$\dot{q} = [\dot{\theta}_1, \cdots, \dot{\theta}_N]$$

Proof. See (P.2), (P.3a) and (V.4). QED

- Per (P.4) and (10), the system mass matrix is

$$\mathcal{M} = G^T \Psi_D^T M \Psi_D G \qquad \text{(P.4a)}$$

- Proposition 5. Given the momentum state, p, the system rate state is

$$\dot{q} = \mathcal{M}^{-1} p \tag{P.5a}$$

$$= [e - K^T \Phi_D \Psi_\beta G] R^{-1} [e - G^T \Psi_\beta^T \Phi_D^T K] p \tag{P.5b}$$

(See [1][2] on the factorization of $\mathcal{M}^{-1}$. Note that the $\mathcal{E}$ operator in [1] is the Φ operator here.)

where $\dot{q} = [\dot{\theta}_i]_{i=1:N}$

$\beta = diag[\left(e - G_i K_i^T\right) D_i]_{i=1:N}$

$D = diag[D_i]_{i=1:N}$, with $D_1 = 0_{6\times 6}$

$R^{-1} = diag[R_i^{-1}]_{i=1:N}, R_i = G_i^T J_i G_i$

$J_i =$ $branch_i$ inertia tensor

$K = diag[K_i]_{i=1:N}, K_i = J_i G_i R_i^{-1}$

$G = diag[G_i]_{i=1:N}, G_i = [g_i, 0_{3\times 1}]_{i=1:N}$

- Backward recursive computation of branch inertia tensors $\left\{ J_i \right\}_{i=1:N}$:

  a) for $i = 1:N$; $J_i = M_i$; end

  b) for $i = N:2$

  $$\hat{J}_i = J_i - J_i G_i R_i^{-1} G_i^T J_i$$

  $$J_{ip} = J_{ip} + D_i^T \hat{J}_i D_i$$

  end

  (see Ref. [1] and [2].)

- Proposition 5 offers two options to solve $\dot{q}$ given p, see Eqs. (P.5a) and (P.5b).

- Equation (P.5a) requires that $\mathcal{M}$ be setup per (P.4a). See the Newton-Euler Eqs. presentation also. Obtaining $\dot{q}$ by (P.5a) is an O(N^3) process.

- Equation (P.5b) requires that p be left multiplied by the indicated series of operators. All these matrix-vector multiplications are O(N). So, the overall computation by (P.5b) is O(N).

- See 'Order of Algorithms' discussion in Newton-Euler Equations presentation.

Order(N) Numerical Solution

- For $t = \{t_0, t_1, t_2, \cdots t_{end}\}$, do {

  Step 1: Given $[q, p]$ at t_i compute $[\dot{q}, \dot{p}]$:

  a. Compute $\{C_i(\theta_i)\}_{i=1:N}$

  b. Compute $\{I_i, g_i, d_i, s_i, r_i, \eta_i, \overline{y}_i, D_i, G_i, M_i, J_i, K_i, R_i, \beta_i\}_{i=1:N}$

  c. Compute $\dot{q}$ given p; use (P.5b), O(N) option

  d. Compute $\{v_i, \mathscr{K}_i\}_{i=1:N}$; use (V.4) and (P.2)

  e. Compute application specific $\{\hat{\tau}_i, \overline{\tau}_i, \overline{f}_i\}_{i=1:N}$ for $\{\dot{p}_{\theta_i}\}_{i=1:N}$

  f. Assemble $\dot{p}$ per (P.3b)

  Step 2: Given $[q, \dot{q}, p, \dot{p}]$ at t_i, integrate $[q, p]$ to t_{i+1} }

Summary

- The momentum based EOM derived from the Hamilton's equations for a generic tree configured N rigid-body system have been presented.

- Numerical integration of that EOM requires the system rates be solved from the system momenta state. Order(N) and Order(N^3) methods of that rate computation have been presented.

- Momentum based Hamilton's equations are just as powerful and accurate as the Newton Euler equations in defining the dynamics of an N-body system.

References

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