

Generalized Mass Matrix of MBS

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Introduction

- Generalized mass matrix has been an important component in solving the dynamics equations of an N -body mechanism up until 1990's due to the prevailing order(N^3) algorithms. With the rise of order(N) recursive methods of solving the dynamics equations after 2000's, the need for that matrix became somewhat less demanding from a dynamics solving point of view.
- System mass matrix in a control system design is often assumed given or known. The latter is generally false. In those cases, it is important to know how to derive it correctly and efficiently.
- The factored form of the system mass matrix shown here makes it possible to derive its inverse in a factored form. See Refs. [1,2]. This factored inverse mass matrix in turn makes it possible to solve the equations of motion in order(N) manner.

- We shall consider the system mass matrix of a rigid multibody system that has only single axis rotational motion between joint connected bodies. The inboard joint of the root body is connected to the ground. A large group of mechanisms is covered with this joint description that includes robots, mechanical arms, cars, cranes, gimballed antennas, suspension and transmission systems
- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.
- We choose the set of generalized coordinates and system rates for this multibody system as

$$q = \{\theta_i\}_{i=1:N}$$

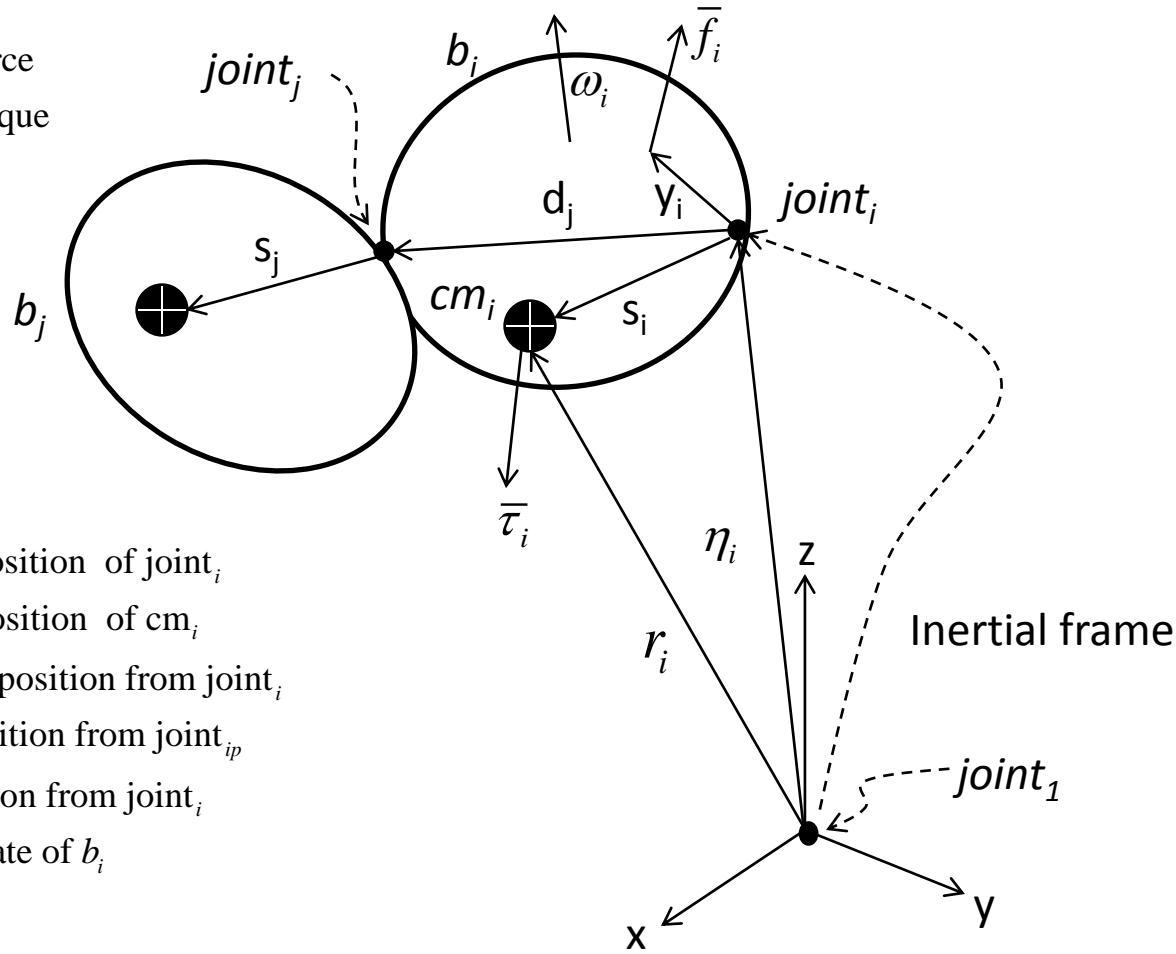
$$\dot{q} = \{\dot{\theta}_i\}_{i=1:N}$$

where θ_i = inboard joint angle of b_i

- From here, we will cover 1) the kinematic equations and the Jacobian, 2) incidence matrix operator, 3) energy equation and 4) generalized mass matrix.

Fig. 1 Notations

\bar{f}_i external force
 $\bar{\tau}_i$ external torque



b_i body i
 η_i inertial position of $joint_i$
 r_i inertial position of cm_i
 y_i \bar{f}_i impact position from $joint_i$
 d_i $joint_i$ position from $joint_{ip}$
 s_i cm_i position from $joint_i$
 ω_i angular rate of b_i

- The root body b_1 is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. In the following, b_1 is connected to the ground
- The chain of bodies between b_1 and b_j shall be denoted as $\{i | i \leq j\}$ or just $i \leq j$. The less-than-or-equal relation over body indices is a topological order and not a numerical order.
- The set of bodies branching from b_j shall be denoted as $\{i | i \geq j\}$ or just $i \geq j$. The greater-than-or-equal relation over body indices is a topological order and not a numerical order.
- All vectors in the following discussion are given in the format x_j^i . The subscript j denotes the body that x belongs to and the superscript i denotes the coordinate frame that the vector is in.
- Vectors with no superscript are given in inertial coordinates unless defined otherwise

- A stacked vector y is a column vector whose elements are also vectors. The latter can be of different sizes. We define this stacked vector as

$$y = [y_1, y_2, y_3, \dots, y_N], y_i \text{ is a vector}$$

$$\text{where } \text{size}(y) = \text{len}(y) \times 1, \text{len}(y) = \sum_{i=1}^N \text{len}(y_i)$$

For example:

$$\text{Let } y_1 = v_x, y_2 = \begin{bmatrix} s_x \\ s_y \end{bmatrix}, y_3 = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$\text{Then } y = [y_1, y_2, y_3] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} v_x \\ s_x \\ s_y \\ w_x \\ w_y \\ w_z \end{bmatrix}$$

- Skew matrix notation: $\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$, if $a = [a_x, a_y, a_z]$
- $e = k \times k$ identity matrix, size k depends on context
- The relations $\{ >, <, \geq, \leq \}$ shall mean topological order in the following expressions when used to group bodies, unless stated otherwise.
- $\bar{i} = \{ \alpha \mid \text{parent}(\alpha) = i \}$, indices of children of b_i

Influence Matrices $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}^T$

- Influence matrices, $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}^T$, are defined by the parent child relations of the considered mechanism and a block diagonal matrix \mathcal{A} :

$$[\Phi_{\mathcal{A}}]_{i,j} = \begin{cases} \mathcal{A}_i & \text{if } b_j = \text{parent of } b_i \\ 0_{6 \times 6} & \text{otherwise} \end{cases} \quad \text{Def. 1}$$

$$[\Phi_{\mathcal{A}}^T]_{i,j} = \begin{cases} \mathcal{A}_j^T & \text{if } b_i = \text{parent of } b_j \\ 0_{6 \times 6} & \text{otherwise} \end{cases} \quad \text{Def. 2}$$

where $\mathcal{A} = \text{diag}\{\mathcal{A}_i\}_{i=1:N}$, a block diagonal matrix, with $\mathcal{A}_i \in \mathbb{R}^{k \times k}$, $\mathcal{A}_1 = 0$
 $k = \text{integer} > 1$

- Forward influence matrix, $\Phi_{\mathcal{A}}$ is strictly lower triangular.
- Backward influence matrix, $\Phi_{\mathcal{A}}^T$ is strictly upper triangular.

- Each body in a tree-configured mechanism has one parent body except for b_1 . Thus, each row of $\Phi_{\mathcal{A}}$ has only one non-zero submatrix entry and the first row is all zeros.

- Given that $\Phi_{\mathcal{A}}$ is square and strictly triangular, it is nilpotent of degree m , i.e.

$$\Phi_{\mathcal{A}}^m = 0$$

where, $m = \text{length of the longest link in the system from } b_1$, and $m \leq N$.

- Let $\mathcal{A} = \text{diag}\{\mathcal{A}_i\}_{i=1:N}$, $\mathcal{A}_i \in R^{k \times k}$, $\mathcal{A}_1 = 0$, and $x = \text{col}[x_i]$, $z = \text{col}[z_i]$, with $x_i, z_i \in R^k$, $k > 1$.

Prop1. $z = \Phi_{\mathcal{A}}x$ is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = \mathcal{A}_i x_{ip} \text{ for } i = 1:N \quad (1)$$

Prop2. $z = \Phi_{\mathcal{A}}^T x$ is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j \in \bar{i}} \mathcal{A}_j^T x_j \text{ for } i = 1:N \quad (2)$$

- If $\Phi_{\mathcal{A}}$ is square and strictly lower triangular, then $\Psi_{\mathcal{A}} = (e - \Phi_{\mathcal{A}})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}} = e + \Phi_{\mathcal{A}} + \Phi_{\mathcal{A}}^2 + \dots + \Phi_{\mathcal{A}}^{m-1}$$

where $m =$ nilpotency of $\Phi_{\mathcal{A}}$

- It follows that $\Phi_{\mathcal{A}}^T$ is square and strictly upper triangular, and $\Psi_{\mathcal{A}}^T = (e - \Phi_{\mathcal{A}}^T)^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}}^T = e + \Phi_{\mathcal{A}}^T + \Phi_{\mathcal{A}}^{T,2} + \dots + \Phi_{\mathcal{A}}^{T,m-1}$$

- The power series expansion of $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}^T$ serve to show the existence of $(e - \Phi_{\mathcal{A}})^{-1}$ and $(e - \Phi_{\mathcal{A}}^T)^{-1}$. More efficient way of computing $\Psi_{\mathcal{A}}x$ and $\Psi_{\mathcal{A}}^Tx$ for any $x \in \mathbb{R}^{6N}$ is shown shortly.

$O(N)$ Recursive Operators

- The operator $\Psi_{\mathcal{A}}$ is a function of the given system parent-child relations embedded in $\Phi_{\mathcal{A}}$.
- Computing expressions involving $(\Psi_{\mathcal{A}}x)$ or $(\Psi_{\mathcal{A}}^T x)$ by direct matrix-vector multiplication is not efficient.
- The next two procedures to compute the two operations are order(N).

Algorithm for $z = \Psi_{\mathcal{A}}x$

- A stacked vector given by $z = \Psi_{\mathcal{A}}x$ is the result of solving $z = \Phi_{\mathcal{A}}z + x$ for z , where $\Phi_{\mathcal{A}}$ is square and strictly lower triangular per Def 1. Thus, given x , the elements of z are obtained by the algorithm below :

1. set $z_1 = x_1$ since the first row of $\Phi_{\mathcal{A}}$ is zero

2. for $i = 2 : N$

$$z_i = \mathcal{A}_i z_{i_p} + x_i \quad ;$$

end

- $\Psi_{\mathcal{A}}$ is an order(N) operator, since $z = \Psi_{\mathcal{A}}x$ is computed in one ' N do-loop'.

Algorithm for $z = \Psi_{\mathcal{A}}^T x$

- A stacked vector given by $z = \Psi_{\mathcal{A}}^T x$ is the result of solving $z = \Phi_{\mathcal{A}}^T z + x$ for z , where $\Phi_{\mathcal{A}}^T$ is square and strictly upper triangular. Thus, given x , the elements of z are obtained by the algorithm below :

1. set $z = x$ since the first column of $\Phi_{\mathcal{A}}^T$ is zero

2. for $i = N : 2$

$$z_{ip} := z_{ip} + \mathcal{A}_i^T z_i \quad ;$$

end

- $\Psi_{\mathcal{A}}^T$ is an order(N) operator, since $z = \Psi_{\mathcal{A}}^T x$ is computed in one ' N do-loop'.

- In light of Def. 1 and Prop. 1, the joint velocity vectors can be written in a stacked vector form as

$$v = \Phi_D v + G \dot{q} \in \mathbb{R}^{6N \times 1} \quad (3)$$

where $v = [v_1, v_2, v_3, \dots, v_N]$

$v_i = [\omega_i, \dot{\eta}_i]$, joint velocity pair for b_i , see Figure 1

Φ_D = incidence matrix given the D matrix

$$D = \text{diag}\{D_i\}_{i=1:N}, D_1 = 0_{6 \times 6}, D_i = \begin{bmatrix} e_{3 \times 3} & 0 \\ -\tilde{d}_i & e_{3 \times 3} \end{bmatrix}_{i=2:N}$$

e = identity matrix

$$G = \text{diag}\{G_i\}_{i=1:N}, G_i = [g_i, 0_{3 \times 1}] \in \mathbb{R}^{6 \times 1}$$

$g_i \in \mathbb{R}^{3 \times 1}$, free motion axis of joint _{i}

$\dot{q} = [\dot{\theta}_1, \dots, \dot{\theta}_N]$, rate state

- The stacked joint velocity vector in Eq. (3) can be solved as

$$v = \Psi_D G \dot{q} \quad (4)$$

where $\Psi_D = (e - \Phi_D)^{-1}$, $e = 6N \times 6N$ identity matrix

- The body velocities $[\bar{v}_i]_{i=1:N}$ relate to the joint velocities $[v_i]_{i=1:N}$ in a stacked vector form as

$$\bar{v} = S v \in \mathbf{R}^{6N \times 1} \quad (5)$$

where $\bar{v} = [\omega_i, \dot{r}_i]_{i=1:N}$, body velocity pairs

$v = [\omega_i, \dot{\eta}_i]_{i=1:N}$, joint velocity pairs

$S = \text{diag}[S_i]_{i=1:N}$

$$S_i = \begin{bmatrix} e_{3 \times 3} & 0 \\ -\tilde{s}_i & e_{3 \times 3} \end{bmatrix}$$

$s_i = \text{joint}_i$ to cm_i displacement vector in inertial coordinates, see Fig. 1

- Given Eqs. (4) and (5), the body rate vectors relate to the rate state, \dot{q} , as

$$\bar{v} = S\Psi_D G\dot{q} \tag{6}$$

- Thus the Jacobian of the body velocities defined by $\frac{\partial \bar{v}}{\partial \dot{q}}$ is

$$J(q) = \frac{\partial \bar{v}}{\partial \dot{q}} = S\Psi_D G \in \mathbf{R}^{6N \times N} \tag{7}$$

Kinetic Energy and Mass Matrix

- Kinetic energy of an N -body system is

$$T = \frac{1}{2} \sum_{i=1}^N \int_{b_i} (\dot{l} + \dot{r}_i)^T (\dot{l} + \dot{r}_i) dm \quad (8)$$

where l = displacement of dm from cm_i

$$\dot{l} = \omega_i \times l$$

- Equation (8) reduces to

$$T = \frac{1}{2} \sum_{i=1}^N (\omega_i^T I_i \omega_i + m_i \dot{r}_i^T \dot{r}_i) \quad (9)$$

where $I_i = -\int_{b_i} \tilde{l} \tilde{l} dm$, $\int_{b_i} l dm = 0$, $m_i = \int_{b_i} dm$

- Eq. (9) reduces to

$$T = \frac{1}{2} \sum_{i=1}^N \bar{\mathbf{v}}_i^T \bar{\mathbf{M}}_i \bar{\mathbf{v}}_i \quad (10)$$

where $\bar{\mathbf{M}}_i = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & m_i \mathbf{e} \end{bmatrix}, \bar{\mathbf{v}}_i = [\boldsymbol{\omega}_i, \dot{\mathbf{r}}_i]$

- By stacking the body velocity vectors in Eq. (10), the latter becomes

$$T = \frac{1}{2} \bar{\mathbf{v}}^T \bar{\mathbf{M}} \bar{\mathbf{v}} \quad (11)$$

where $\bar{\mathbf{M}} = \text{diag}[\bar{\mathbf{M}}_i]_{i=1:N} \in \mathbf{R}^{6N \times 6N}, \bar{\mathbf{v}} = [\bar{\mathbf{v}}_i]_{i=1:N}$

- By Eq. (6) the kinetic energy becomes

$$\begin{aligned} T &= \frac{1}{2} \dot{q}^T G^T \Psi_D^T S^T \bar{M} S \Psi_D G \dot{q} \\ &= \frac{1}{2} \dot{q}^T G^T \Psi_D^T M \Psi_D G \dot{q} \end{aligned} \quad (12)$$

where $M = \text{diag}[M_i]_{i=1:N} \in R^{6N \times 6N}$

$$M_i = S_i^T \bar{M}_i S_i = \begin{bmatrix} I_i - m_i \tilde{s}_i \tilde{s}_i & m_i \tilde{s}_i \\ -m_i \tilde{s}_i & m_i e \end{bmatrix} \in R^{6 \times 6}$$

- Eq. (12) simplifies to

$$T = \frac{1}{2} \dot{q}^T \mathcal{M} \dot{q} \quad (13)$$

with $\mathcal{M} = G^T \Psi_D^T M \Psi_D G \in R^{N \times N}$ (14)

- The second partial derivative of T w.r.t. \dot{q} per Eq. (13) yields the system mass matrix as

$$\mathcal{M} = \frac{\partial^2 T}{\partial^2 \dot{q}} \quad (15)$$

Computing System Mass Matrix

Procedure to compute \mathcal{M} per (14):

1. Compute $J = \Psi_D G$ \Rightarrow 2. Compute $Q = MJ$
with $J = [J(:,1) J(:,2) \cdots J(:,N)]$ with $Q = [Q(:,1) Q(:,2) \cdots Q(:,N)]$
 $G = [G(:,1) G(:,2) \cdots G(:,N)]$ $M = \text{diag}\{M(k), k = 1:N\}$
for $i = 1:N$ for $i = 1:N$
 $J(:,i) = \Psi_D G(:,i)$; per Alg.1 $Q(:,i) = MJ(:,i)$;
end end

3. $\mathcal{M} = J^T Q$, end

Q1: what is the order of the above procedure?

Q2: can it be made order(N)?

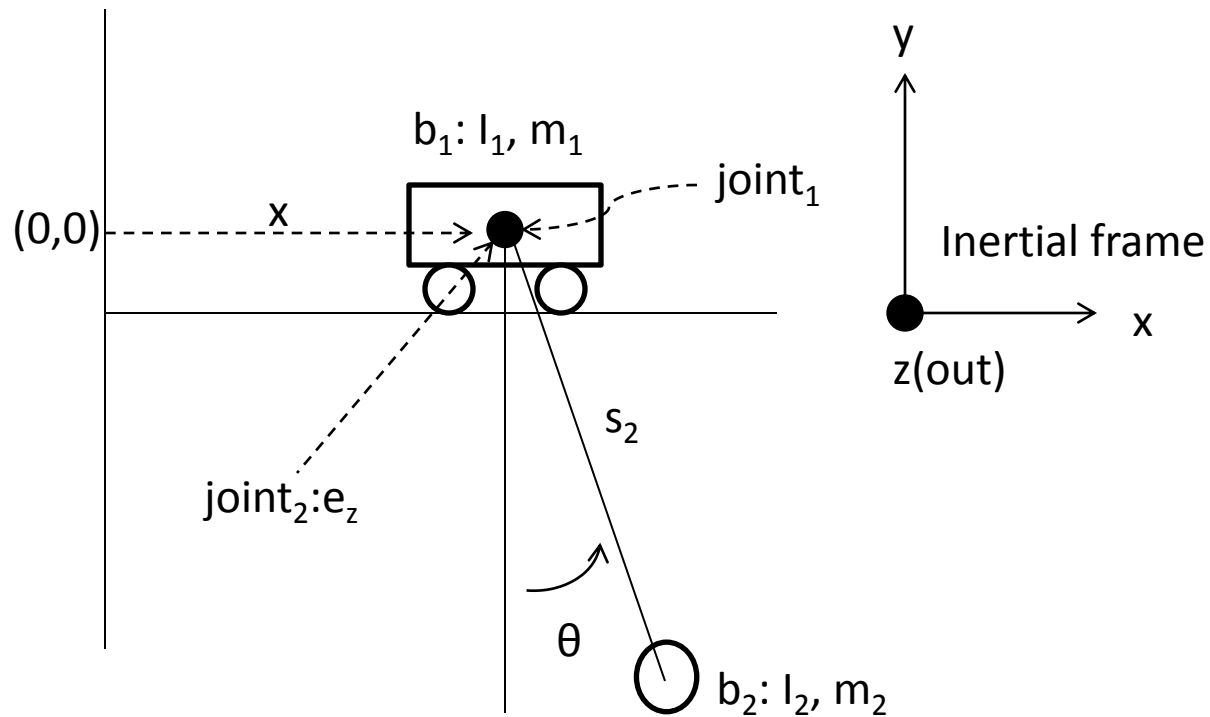
- Ans1: Steps 1, 2 and 3 are each order(N^2), therefore the procedure is order(N^2)
- Ans2: No. Computation of \mathcal{M} is order(N^2) per Ans1.

Examples

- Each system mass matrix of the four examples that follow is hand derived to illustrate the explicit expression of the elements of the matrix for the model of interest.
- For practical purposes use the given system mass matrix procedure to compute the numerical value of that matrix.
- Test the procedure-based matrix result against the values of hand derived system mass matrix to verify that they are identical.

Example 1

Pendulum with a Moving Base



Model Parameters

$$D = \text{diag}\{0_{6 \times 6}, D_2\}, \text{ with } D_2 = e_{6 \times 6}, \text{ given } d_2 = 0, \omega_1 = 0, \eta_1 = xe_x \\ s_1 = 0, s_2 = L(\sin(\theta)e_x - \cos(\theta)e_y)$$

$$G = \text{diag}\{G_1, G_2\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}_{12 \times 2}, \text{ with } G_1 = \begin{bmatrix} 0 \\ e_x \end{bmatrix}, G_2 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \quad (\text{M1})$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, 0 = 0_{6 \times 6} \text{ nilpotency}(\Phi_D) = 2 \quad (\text{M2})$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6 \times 6} & 0 \\ e_{6 \times 6} & e_{6 \times 6} \end{bmatrix} \quad (\text{M3})$$

$$M = \text{diag}\{M_1, M_2\} \quad (\text{M4})$$

- Rate state for this example is $\dot{q} = [\dot{x}, \dot{\theta}_2]$

- Given (M1) and (M3), we have

$$\Psi_D G = \begin{bmatrix} G_1 & 0 \\ G_1 & G_2 \end{bmatrix}_{12 \times 2} \quad (\text{M5})$$

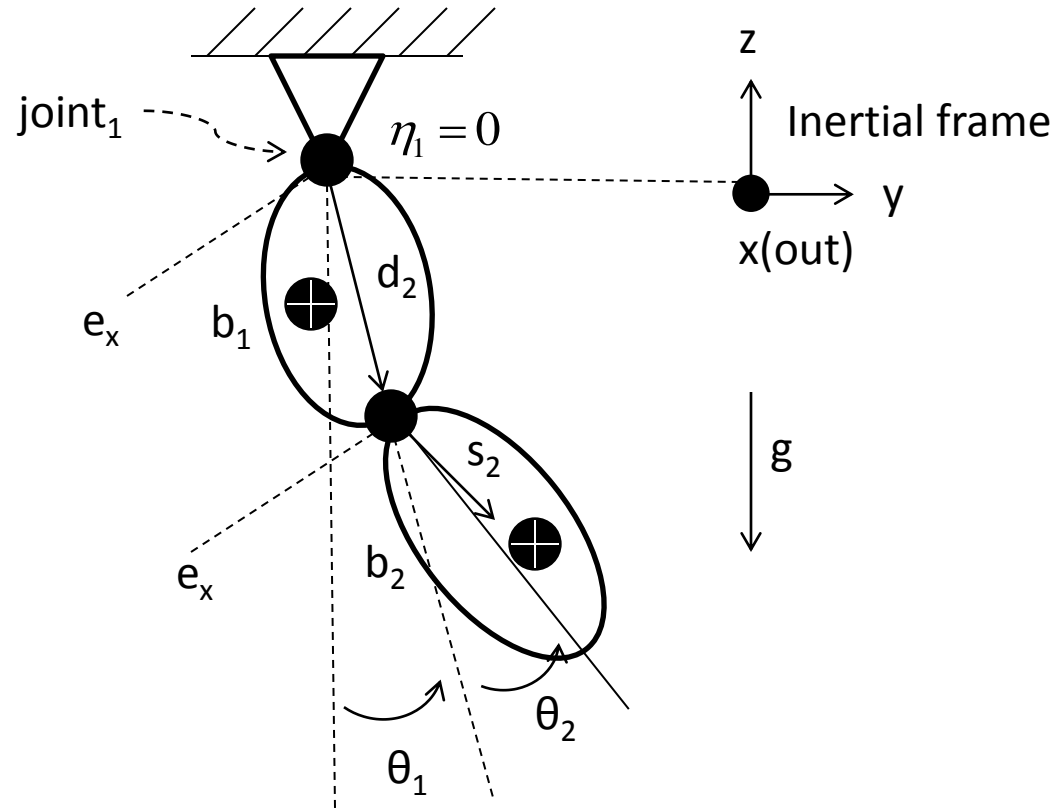
- The system mass matrix of the pendulum on a moving base per (13) is

$$\begin{aligned} \mathcal{M} &= G^T \Psi_D^T M \Psi_D G \\ &= \begin{bmatrix} G_1^T & G_1^T \\ 0 & G_2^T \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ G_1 & G_2 \end{bmatrix} \\ &= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T M_2 G_1 & G_1^T M_2 G_2 \\ G_2^T M_2 G_1 & G_2^T M_2 G_2 \end{bmatrix} \\ &= \begin{bmatrix} [0, e_x]^T M_1 [0, e_x] + [0, e_x]^T M_2 [0, e_x] & [0, e_x]^T M_2 [e_z, 0] \\ [e_z, 0]^T M_2 [0, e_x] & [e_z, 0]^T M_2 [e_z, 0] \end{bmatrix} \\ &= \begin{bmatrix} m_1 + m_2 & m_2 L \cos(\theta) \\ m_2 L \cos(\theta) & I_2^{zz} + m_2 L^2 \end{bmatrix}_{2 \times 2} \end{aligned}$$

QED

Example 2

Double Pendulum



Model Parameters

$$D = \text{diag}\{0_{6 \times 6}, D_2\}, \text{ with } d_1 = 0, D_2 = \begin{bmatrix} e & 0 \\ -\tilde{d}_2 & e \end{bmatrix}, \text{ with } \eta_1 = 0$$

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}_{12 \times 2}, \text{ with } G_1 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1} \quad (\text{N1})$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix}, \text{ nilpotency}(\Phi_D) = 2 \quad (\text{N2})$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6 \times 6} & 0 \\ D_2 & e_{6 \times 6} \end{bmatrix} \quad (\text{N3})$$

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}_{12 \times 12} \quad (\text{N4})$$

- Rate state for this example is $\dot{q} = [\dot{\theta}_1, \dot{\theta}_2]$

- By (N1) and (N3) we have

$$\Psi_D G = \begin{bmatrix} G_1 & 0 \\ D_2 G_1 & G_2 \end{bmatrix}_{12 \times 2} \quad (\text{N5})$$

- The system mass matrix of the double pendulum per (13) is

$$\begin{aligned} \mathcal{M} &= G^T \Psi_D^T M \Psi_D G \\ &= \begin{bmatrix} G_1^T & G_1^T D_2^T \\ 0 & G_2^T \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ D_2 G_1 & G_2 \end{bmatrix} \\ &= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T D_2^T M_2 D_2 G_1 & G_1^T D_2^T M_2 G_2 \\ G_2^T M_2 D_2 G_1 & G_2^T M_2 G_2 \end{bmatrix} \\ &= \begin{bmatrix} [e_x, 0]^T M_1 [e_x, 0] + [e_x, -\tilde{d}_2 e_x]^T M_2 [e_x, -\tilde{d}_2 e_x] & [e_x, -\tilde{d}_2 e_x]^T M_2 [e_x, 0] \\ [e_x, 0]^T M_2 [e_x, -\tilde{d}_2 e_x] & [e_x, 0]^T M_2 [e_x, 0] \end{bmatrix}_{2 \times 2} \end{aligned} \quad (\text{N6})$$

- Equation (N6) can be expanded with some algebraic manipulations to be

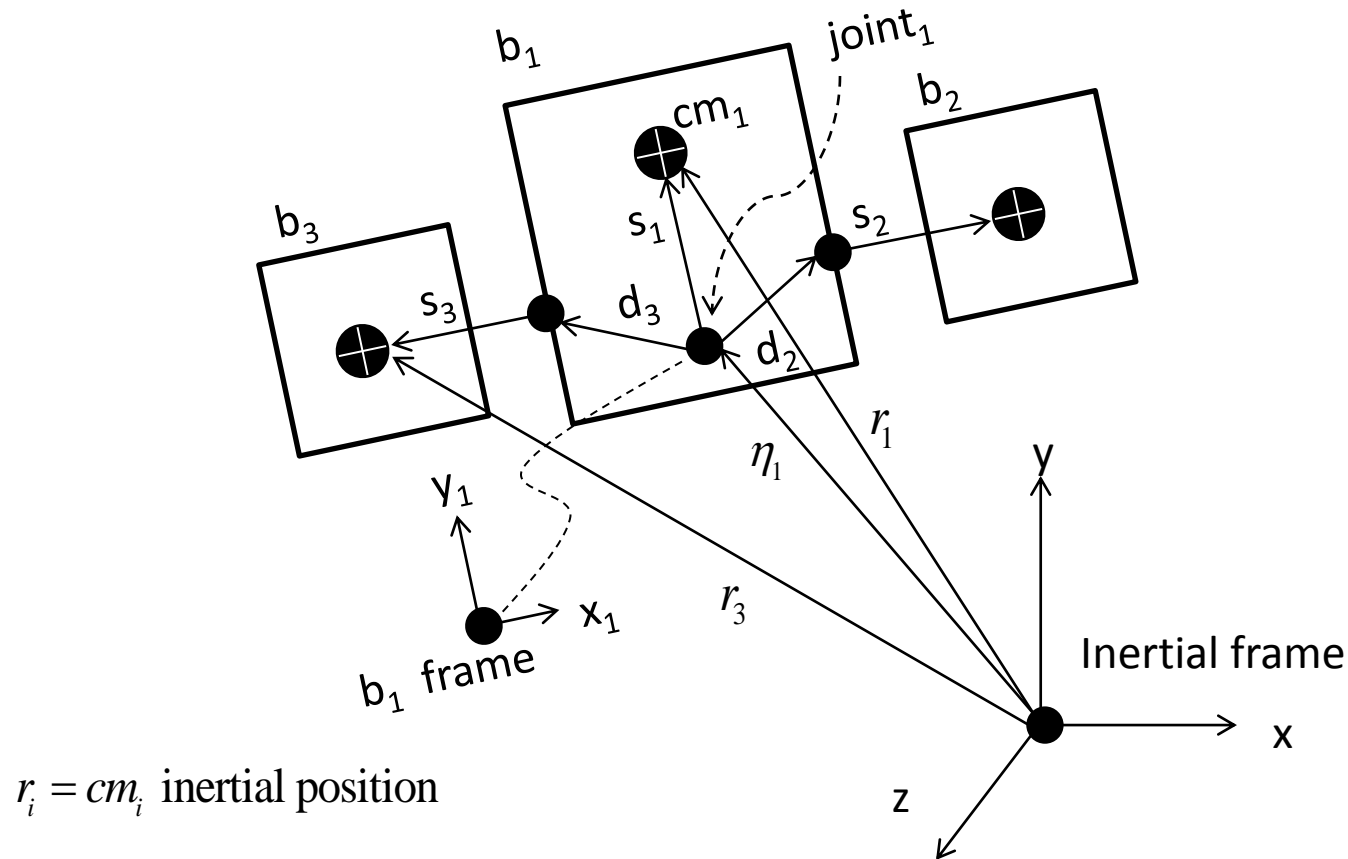
$$\mathcal{M} = \begin{bmatrix} e_x^T (I_1 + I_2 - m_1 \tilde{r}_1 \tilde{r}_1 - m_2 \tilde{r}_2 \tilde{r}_2) e_x & e_x^T (I_2 - m_2 \tilde{r}_2 \tilde{s}_2) e_x \\ e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{r}_2) e_x & e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{s}_2) e_x \end{bmatrix}$$

where $r_1 = s_1$, $r_2 = d_2 + s_2$

QED

Example 3

A Satellite with Two Arrays



Model Parameters

$$D = \text{diag}\{0_{6 \times 6}, D_2, D_3\}, \text{ with } D_j = \begin{bmatrix} e_{3 \times 3} & 0 \\ -\tilde{d}_j & e_{3 \times 3} \end{bmatrix}_{j=2,3} \quad (\text{O1})$$

$$e_x = C_1^{\text{in}} [1, 0, 0] \quad (\text{O2})$$

$$G = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}_{18 \times 8}, \text{ with } G_1 = e_{6 \times 6}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1}, G_3 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1} \quad (\text{O3})$$

$$\Phi_D = \begin{bmatrix} 0 & 0 & 0 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{bmatrix}_{18 \times 18} \text{ with } 0 = 0_{6 \times 6}, \text{ nilpotency}(\Phi_D) = 2 \quad (\text{O4})$$

$$\Psi_D = (e - \Phi_D)^{-1} = (e + \Phi_D) = \begin{bmatrix} e & 0 & 0 \\ D_2 & e & 0 \\ D_3 & 0 & e \end{bmatrix}_{18 \times 18} \quad (\text{O5})$$

- Rate state for this satellite is $\dot{q} = [v_1, \dot{\theta}_1, \dot{\theta}_2]$ with $v_1 = [\omega_1, \dot{\eta}_1]$
- Given (O3) and (O5), we have

$$\Psi_D G = \begin{bmatrix} e & 0 & 0 \\ D_2 & G_2 & 0 \\ D_3 & 0 & G_3 \end{bmatrix}_{18 \times 8} \quad (O6)$$

- The system mass matrix per (13) is

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} e & D_2^T & D_3^T \\ 0 & G_2^T & 0 \\ 0 & 0 & G_3^T \end{bmatrix} \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ D_2 & G_2 & 0 \\ D_3 & 0 & G_3 \end{bmatrix} \\ &= \begin{bmatrix} M_1 + D_2^T M_2 D_2 + D_3^T M_3 D_3 & D_2^T M_2 G_2 & D_3^T M_3 G_3 \\ G_2^T M_2 D_2 & G_2^T M_2 G_2 & 0 \\ G_3^T M_3 D_3 & 0 & G_3^T M_3 G_3 \end{bmatrix}_{8 \times 8} \quad (O7) \end{aligned}$$

- Given (O1) to (O3), and after some algebraic manipulations, (O7) becomes

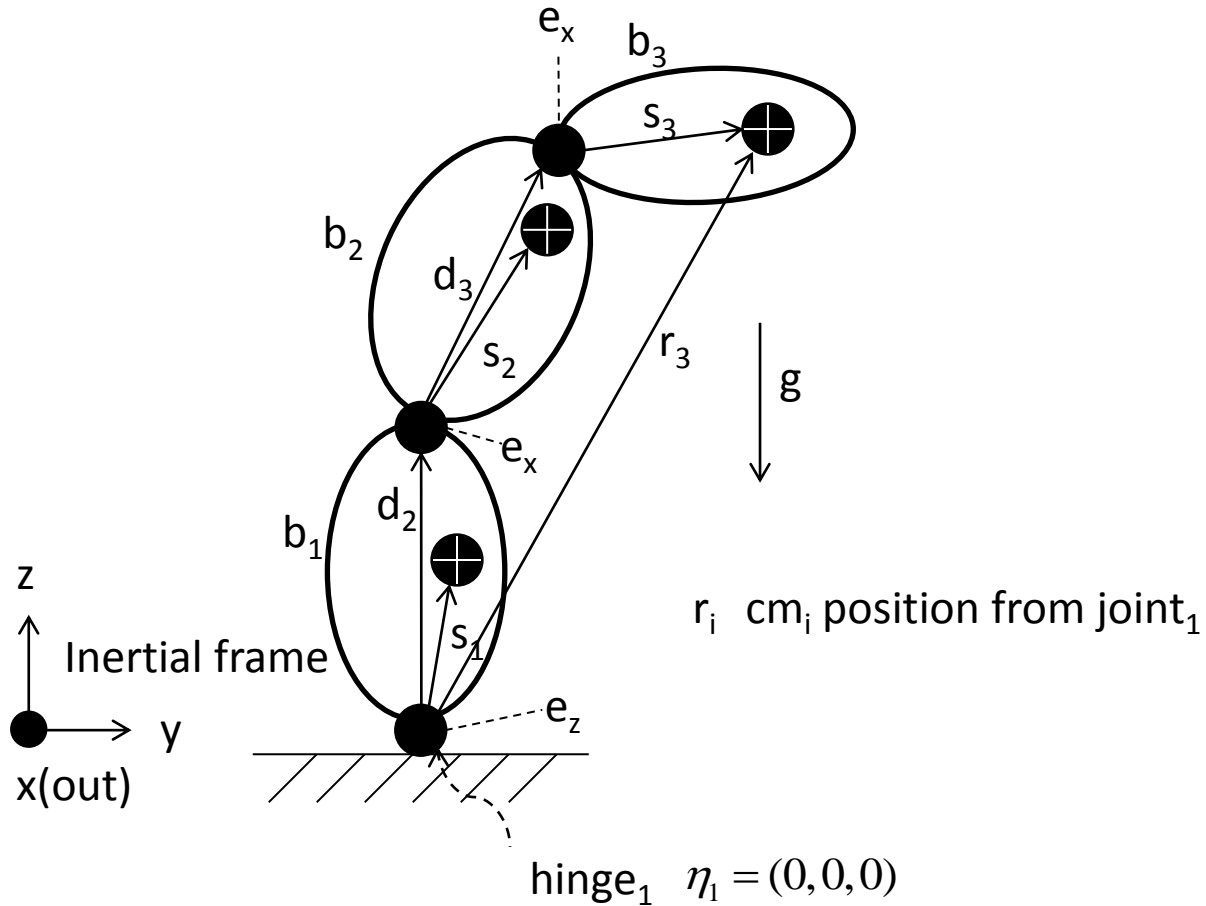
$$\mathcal{M} = \begin{bmatrix} \sum_{j=1}^3 (I_j - m_j \tilde{r}_j \tilde{r}_j) & \bar{m} \tilde{c} & (I_2 - m_2 \tilde{r}_2 \tilde{s}_2) e_x & (I_3 - m_3 \tilde{r}_3 \tilde{s}_3) e_x \\ -\bar{m} \tilde{c} & \bar{m} e & -m_2 \tilde{s}_2 e_x & -m_3 \tilde{s}_3 e_x \\ e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{r}_2) & e_x^T m_2 \tilde{s}_2 & e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{s}_2) e_x & 0 \\ e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{r}_3) & e_x^T m_3 \tilde{s}_3 & 0 & e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{s}_3) e_x \end{bmatrix}$$

where $r_1 = s_1, r_j = d_j + s_j$ for $j = 2:3$, $\bar{m} = \sum_{j=1}^3 m_j$, $c = \frac{\sum_{j=1}^3 m_j r_j}{\bar{m}}$

QED

Example 4

3-Link Arm



Model Parameters

$$D = \text{diag}\{0_{6 \times 6}, D_2, D_3\}, D_j = \begin{bmatrix} e & 0 \\ -\tilde{d}_j & e \end{bmatrix}_{j=2,3} \quad \text{and } \eta_1 = 0$$

$$G = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}_{18 \times 3}, \quad \text{with } G_1 = \begin{bmatrix} e_z \\ 0 \end{bmatrix}_{6 \times 1}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1}, G_3 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6 \times 1} \quad (\text{P1})$$

$$\Phi_D = \begin{bmatrix} 0 & 0 & 0 \\ D_2 & 0 & 0 \\ 0 & D_3 & 0 \end{bmatrix}_{18 \times 18}, \quad \text{nilpotency}(\Phi_D) = 3 \quad (\text{P2})$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D + \Phi_D^2 = \begin{bmatrix} e_{6 \times 6} & 0 & 0 \\ D_2 & e_{6 \times 6} & 0 \\ D_3 D_2 & D_3 & e_{6 \times 6} \end{bmatrix} \quad (\text{P3})$$

$$M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}_{18 \times 18} \quad (\text{P4})$$

- Generalized rates for this arm is $\dot{q} = [\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]$

- Given Eqs. (P1) and (P2), we have

$$\Psi_D G = \begin{bmatrix} G_1 & 0 & 0 \\ D_2 G_1 & G_2 & 0 \\ D_{3:2} G_1 & D_3 G_2 & G_3 \end{bmatrix}_{18 \times 3} \quad (\text{P5})$$

where $D_{3:2} = D_3 D_2 = \begin{bmatrix} e & 0 \\ -(d_3 + d_2)^* & e \end{bmatrix}_{6 \times 6}$

- System mass matrix of the 3-link arm is

$$\mathcal{M} = G^T \Psi_D^T M \Psi_D G$$

$$\begin{aligned} &= \begin{bmatrix} G_1^T & G_1^T D_2^T & G_1^T D_{3:2}^T \\ 0 & G_2^T & G_2^T D_3^T \\ 0 & 0 & G_3^T \end{bmatrix} \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} G_1 & 0 & 0 \\ D_2 G_1 & G_2 & 0 \\ D_{3:2} G_1 & D_3 G_2 & G_3 \end{bmatrix} \\ &= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T D_2^T M_2 D_2 G_1 + G_1^T D_{3:2}^T M_3 D_{3:2} G_1 & G_1^T D_2^T M_2 G_2 + G_1^T D_{3:2}^T M_3 D_3 G_2 & G_1^T D_{3:2}^T M_3 G_3 \\ G_2^T M_2 D_2 G_1 + G_2^T D_3^T M_3 D_{3:2} G_1 & G_2^T M_2 G_2 + G_2^T D_3^T M_3 D_3 G_2 & G_2^T D_3^T M_3 G_3 \\ G_3^T M_3 D_{3:2} G_1 & G_3^T M_3 D_3 G_2 & G_3^T M_3 G_3 \end{bmatrix}_{3 \times 3} \quad (\text{P6}) \end{aligned}$$

- Equation (P6) can be expanded to be

$$\mathcal{M} = \begin{bmatrix} e_z^T \sum_{j=1}^3 (I_j - m_j \tilde{r}_j \tilde{r}_j) e_z & e_z^T \sum_{j=2}^3 (I_j - m_j \tilde{r}_j (\tilde{r}_j - \tilde{\eta}_2)) e_x & e_z^T (I_3 - m_3 \tilde{r}_3 \tilde{s}_3) e_x \\ e_x^T \sum_{j=2}^3 (I_j - m_j (\tilde{r}_j - \tilde{\eta}_2) \tilde{r}_j) e_z & e_x^T \sum_{j=2}^3 (I_j - m_j (\tilde{r}_j - \tilde{\eta}_2) (\tilde{r}_j - \tilde{\eta}_2)) e_x & e_x^T (I_3 - m_3 (\tilde{r}_3 - \tilde{\eta}_2) \tilde{s}_3) e_x \\ e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{r}_3) e_z & e_x^T (I_3 - m_3 \tilde{s}_3 (\tilde{r}_3 - \tilde{\eta}_2)) e_x & e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{s}_3) e_x \end{bmatrix}_{3 \times 3} \quad (\text{P7})$$

where $\eta_1 = 0$, $\eta_2 = d_2$, $\eta_3 = d_2 + d_3$

$r_1 = s_1$, $r_2 = d_2 + s_2$, $r_3 = d_2 + d_3 + s_3$

QED

Summary

- The system mass matrix of a multibody system is shown to be a second order partial derivative of the system kinetic energy with respect to the generalized velocities.
- This mass matrix can be expressed explicitly in a factored form based on the Jacobian matrix of the system body velocities. That expression depends on the influence matrix operator Φ_D and the associated forward and backward recursive operators (Ψ_D, Ψ_D^T) .
- The algorithm presented for computing the system mass matrix is more versatile and efficient than the case-by-case hand derivation of the same matrix for any mechanical system as shown by the examples given.

References

1. Rodriguez, G., Jain, A., Kreutz-Delgado, K., “Spatial Operator Algebra for Manipulator Modeling and Control”, The International Journal of Robotics Research, Vol. 10, No. 4, August, 1991.
2. Tong, M. ,”Inverse Mass Matrix Factorization Using Momentum Equations of a Rigid Multibody System”, AIAA Guidance, Navigation and Control Conference, 13-16 August, 2012, Minneapolis, Minnesota, Paper No. 1359569